In this section...

Definitions

• Hydrostatic balance
• Scale height
• Lagrangian frame
• Eulerian frame
• Material derivative
• Adiabatic motion

• Potential temperature
• Lapse rate
• Lapse rate (dry adiabatic)
• Lapse rate (saturated)
• Foehn wind
Hydrostatic Balance
Parcel Description

Under the parcel dynamics model, the atmosphere is assumed to consist of many distinct air parcels that move around and experience forces.

**Parcel Properties**

- $\rho$  Density
- $p$   Pressure
- $T$  Temperature
- $q_i$ Constituent mixing ratios
- $u$ Velocity vector

Thermodynamic properties related by ideal gas law.
Although the horizontal atmosphere is in a constant state of motion, vertical velocities are typically fairly small (especially averaged over the large scale).

When vertical velocity is approximately zero, density and pressure end up being functionally related.

**Figure:** A vertical column of air of density $\rho$, horizontal cross-section $\delta A$, height $\delta z$ and mass $M = \rho \delta A \delta z$. The pressure at the lower surface is $p$, the pressure at the upper surface is $p + \delta p$. The force at both the lower and upper surfaces is pressure times area.
If the cylinder of air is not accelerating, it must be subject to zero net force. The vertical forces are:

- Pressure acting on bottom face: \( F_B = p\delta A \)
- Pressure acting on top face: \( F_T = -(p + \delta p)\delta A \)
- Gravity: \( F_g = -gM = -g\rho\delta A\delta z \)

**Definition:** A fluid is in **hydrostatic balance** when external forces (here gravity) are balanced by the pressure-gradient force.

\[ (p + \delta p)\delta A \]
Hydrostatic Balance

\[ F_g + F_T + F_B = 0 \]

\[ \delta p + g \rho \delta z = 0 \]

Taylor Series

\[ \delta p \approx \frac{\partial p}{\partial z} \delta z \]

Hydrostatic Balance:

\[ \frac{\partial p}{\partial z} + \rho g = 0 \]

In combination with the ideal gas law, hydrostatic balance enables all thermodynamic variables to be related to temperature and surface pressure.
Hydrostatic Balance

\[ \frac{\partial p}{\partial z} + \rho g = 0 \]

Integrating from the top of the atmosphere downward, and noting that \( p(z = \infty) = 0 \) then gives

\[ p(z) = g \int_{z}^{\infty} \rho \, dz \]

Hence the pressure at a given height level is proportional to the total mass of the atmosphere above it.

\[ p_s = \frac{g M_a}{4\pi a^2} \quad (M_a = \text{total mass of atmosphere}) \]
Hydrostatic Balance

\[ \frac{\partial p}{\partial z} + \rho g = 0 \]

Hydrostatic Balance

This equation does not give pressure explicitly in terms of height, since the density of air is not known.

Ideal gas law

\[ \rho = \frac{p}{RT} \]

Scale height (isothermal)

\[ H = \frac{RT_0}{g} \]

For an isothermal atmosphere \( T = T_0 \)

\[ \frac{\partial p}{\partial z} = -\frac{p}{H} \]

Integrating:

\[ p(z) = p_s \exp \left( -\frac{z}{H} \right) \]
Definition: In general, the **scale height** of the atmosphere is the altitude gain needed for the pressure to decrease by a factor of $e$.

For an isothermal atmosphere $T = T_0$

$$p(z) = p_s \exp \left(-\frac{z}{H}\right)$$

For an isothermal atmosphere with $T_0 = 273K$ the scale height is 8km. This is also the approximate scale height for the troposphere below 10km.

**Scale height (isothermal)**

$$H = \frac{RT_0}{g}$$

**Figure:** Observed profile of pressure (blue) plotted against isothermal profile with $H = 8$km (red).
For an isothermal atmosphere $T = T_0$

$$p(z) = p_s \exp \left( -\frac{z}{H} \right)$$

**Definition:** In general, the **scale height** of the atmosphere is the altitude gain needed for the pressure to decrease by a factor of $e$.

**Scale height (isothermal)**

$$H = \frac{RT_0}{g}$$

For an isothermal atmosphere with $T_0 = 232K$ the scale height is **6.8km**. This is value tends to best capture the pressure below 100km altitude.

**Figure:** Observed profile of pressure (blue) plotted against isothermal profile with $H = 6.8$km (red).
Material Derivative
Think in terms of the components of the atmosphere... Consider the motion of a single fluid parcel:

**Question:** How do we quantify the properties of the global atmosphere?

**Related Question:** How do we quantify the properties of the fluid parcel?
The Lagrangian Frame

**Definition:** In the Lagrangian Frame the properties of the whole atmosphere are described in terms of individual fluid particles (or parcels) whose properties are individually known.

In the **Lagrangian Frame**, properties that are invariant to transport (i.e. mass of air within a parcel) are known immediately.
Weather balloons are a source of measurements in the Lagrangian frame. They are passively transported by the background wind field (and so are essentially isolated fluid parcels).
**Idea:** Measurements could be taken at a single point over a time period. The global atmosphere is then quantified combining these measurements.

**Definition:** In the Eulerian Frame the properties of the whole atmosphere are described in terms of individual measurements taken at fixed locations over time.
Weather stations are a source of measurements in the **Eulerian frame**. They are placed in a fixed location and observe the properties of fluid parcels as they pass by.
An Eulerian Map

Warmest Day of the Year
Day of warmest high temperature based on 1981-2010 climate normals

Eulerian data can be plotted on a gridded map, representing a measurement at each location.
Question: Why consider two frames of reference?

Answer: Since the same physical principles hold regardless of the reference frame, the use of multiple reference frames is primarily for purposes of understanding.

Certain concepts can be more easily explained in the Lagrangian frame, whereas others are better explained in the Eulerian frame.
We are interested in determining a quantitative relationship that connects the Lagrangian and Eulerian frames.

To proceed we need a method to quantify the notion of change in the Lagrangian frame.

**Definition:** The material derivative describes the time rate of change of the physical quantity of some material element (e.g. a fluid parcel) that is subject to a velocity field. It is denoted by

\[
\frac{D}{Dt}
\]
Consider a parcel with some property of the atmosphere, like temperature \(T\), that moves some distance in time \(\Delta t\).

\[
\Delta T = T(t_1, x(t_1)) - T(t_0, x(t_0))
\]
The Material Derivative

We would like to calculate the change in temperature over time $\Delta t$, following the parcel.

Expand the change in temperature in a Taylor series around the temperature at the initial position.

$$\Delta T = \frac{\partial T}{\partial t} \Delta t + \frac{\partial T}{\partial x} \Delta x + \frac{\partial T}{\partial y} \Delta y + \frac{\partial T}{\partial z} \Delta z + \text{Higher Order Terms}$$

Assume increments over $\Delta t$ are small, and ignore Higher Order Terms.
The Material Derivative

\[ \Delta T = \frac{\partial T}{\partial t} \Delta t + \frac{\partial T}{\partial x} \Delta x + \frac{\partial T}{\partial y} \Delta y + \frac{\partial T}{\partial z} \Delta z \]

Divide through by \( \Delta t \)

\[ \frac{\Delta T}{\Delta t} = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x} \frac{\Delta x}{\Delta t} + \frac{\partial T}{\partial y} \frac{\Delta y}{\Delta t} + \frac{\partial T}{\partial z} \frac{\Delta z}{\Delta t} \]

Take the limit for small \( \Delta t \)

\[ \frac{DT}{Dt} = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x} \frac{Dx}{Dt} + \frac{\partial T}{\partial y} \frac{Dy}{Dt} + \frac{\partial T}{\partial z} \frac{Dz}{Dt} \]
The Material Derivative

Remember, by definition:

\[
\frac{Dx}{Dt} = u, \quad \frac{Dy}{Dt} = v, \quad \frac{Dz}{Dt} = w
\]

... so the material derivative becomes

\[
\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z}
\]

Lagrangian Frame

Eulerian Frame

Change due to advection

Coordinate independent form
This formula connects the Lagrangian frame (which describes the properties of fluid parcels) and the Eulerian frame (which describes the properties at a particular location).
The Material Derivative

**Question:** What is the change in temperature at a point?

\[
\frac{\partial T}{\partial t} = \frac{DT}{Dt} - \mathbf{u} \cdot \nabla T
\]
Question: What is the change in temperature at a point?

\[ \frac{\partial T}{\partial t} = \frac{D T}{D t} - \mathbf{u} \cdot \nabla T \]
Advection: This is the same as taking the derivative of $T$ along the line defined by the velocity vector and multiplying by (-1).

$$\frac{\partial T}{\partial t} = \frac{DT}{Dt} - \mathbf{u} \cdot \nabla T$$

Question: What is the change in temperature at a point?
The Material Derivative

\[
\frac{\partial T}{\partial t} = \frac{DT}{Dt} - \mathbf{u} \cdot \nabla T
\]

Define a new coordinate \( s \) in the direction of \( \mathbf{u} \). Then:

\[-\mathbf{u} \cdot \nabla T = \frac{\partial T}{\partial s}\]

Observe: If \( \frac{dT}{ds} \) is positive then advection causes Eulerian temperature to decrease.
Adiabatic Transport
Recall the first law of thermodynamics:

\[ dU = \delta W + \delta Q \]

With internal energy \( U \), heating \( Q \) and work \( W \)

\[ dU = c_v dT \quad \delta W = -pd\alpha \]

Where \( \alpha = \rho^{-1} \) is the specific volume (volume per unit mass)

First form of the thermodynamic equation:

\[ c_v dT + pd\alpha = \delta Q \]
Thermodynamics

\[ c_v dT + pd\alpha = \delta Q \]

\[ d\alpha = d \left( \frac{1}{\rho} \right) = -\frac{1}{\rho^2} d\rho \]

\[ c_v dT - \frac{p}{\rho^2} d\rho = \delta Q \]

Ideal Gas Law

\[ p = \rho R_d T \]

\[ dp = R_d T d\rho + R_d \rho dT \]

\[ (c_v + R_d) dT - \frac{1}{\rho} d\rho = \delta Q \]
Second form of the thermodynamic equation:

$$c_p dT - \frac{1}{\rho} dp = \delta Q$$

Diabatic heating

**Differential form:** Relates to infinitesimal changes $dT$ and $dp$ and how they are related to one another. Holds in the context of quasi-static equilibrium.

Change in temperature at constant pressure

Work done to change the pressure of the fluid
Second form of the thermodynamic equation for atmospheric fluid parcels:

$$c_p \frac{DT}{Dt} - \frac{1}{\rho} \frac{Dp}{Dt} = J$$

- Change in temperature following a fluid parcel
- Change in pressure following the fluid parcel

Diabatic heating rate
- Units: $J/kg/s$
For most large-scale motions, the amount of latent heating in clouds and precipitation is relatively small.

In absence of sources and sinks of energy \((J = 0)\) entropy is conserved following the motion. Motion of fluid parcels in the absence of external sources and sinks is referred to as adiabatic motion.

Adiabatic motion are particularly relevant for large-scale vertical motion and in the analysis of atmospheric stability.
\[ c_p \frac{DT}{Dt} - \frac{1}{\rho} \frac{DP}{Dt} = J \]

**Ideal gas law**

\[ \rho = \frac{p}{RT} \]

\[ T_0, p_0 \text{ arbitrary} \]

\[ \frac{c_p}{T} \frac{DT}{Dt} - \frac{R}{p} \frac{DP}{Dt} = \frac{J}{T} \]

\[ c_p \frac{D}{Dt} \log\left(\frac{T}{T_0}\right) - R \frac{D}{Dt} \log\left(\frac{p}{p_0}\right) = \frac{J}{T} \]

\[ \frac{D}{Dt} \log\left(\frac{T}{T_0}\right) - \frac{R}{c_p} \frac{D}{Dt} \log\left(\frac{p}{p_0}\right) = \frac{J}{c_p T} \]

\[ \frac{D}{Dt} \log \left[ \frac{T}{T_0} \left(\frac{p_0}{p}\right)^{R/c_p} \right] = \frac{J}{c_p T} \]
Thermodynamics

\[ \frac{D}{Dt} \log \left[ \frac{T}{T_0} \left( \frac{p_0}{p} \right)^{R/c_p} \right] = \frac{J}{c_p T} \]

**Potential Temperature**

\[ \theta = T \left( \frac{p_0}{p} \right)^{R/c_p} \]

This is also referred to as Poisson’s equation.

\[ \frac{D}{Dt} \log \left( \frac{\theta}{T_0} \right) = \frac{J}{c_p T} \]

If the flow is adiabatic, this equation reduces to:

\[ \frac{D\theta}{Dt} = 0 \]

(potential temperature is conserved following adiabatic flow)

Diabatic heating term
Potential Temperature

\[ \theta = T \left( \frac{p_0}{p} \right)^{R/c_p} \]

In an adiabatic flow:

\[ \frac{D\theta}{Dt} = 0 \]

Typically \( p_0 \) represents sea-level pressure (\( 10^5 \) Pa = 1000 hPa)

In this case, potential temperature is alternatively defined as the temperature an air parcel (with temperature \( T \) and pressure \( p \)) would have if it was adiabatically brought to sea-level pressure.

Potential temperature is closely associated with entropy (constant potential temperature is the same as constant entropy).
Question: The temperature at the top of the continental divide is -10 degrees Celsius (about 263K). The pressure is 600 hPa, $R = 287 \text{ J/kg/K}$, $c_p = 1004 \text{ J/kg/K}$.

(a) What is the potential temperature at the continental divide?

(b) What is the temperature a dry air parcel would have if it sinks to the plains (pressure level of 850 hPa) with no external heating?
Lapse Rate

Below the tropopause the atmosphere cools roughly linearly with altitude. This feature is governed by adiabatic motion and the thermodynamics of air parcels as they are lifted and subside.

**Definition:** The magnitude of the decrease of temperature with height is called the *lapse rate.*
Lapse Rate

For a well-mixed, dry adiabatic, hydrostatically balanced atmosphere the potential temperature $\theta$ does not vary in the vertical direction:

$$\frac{\partial \theta}{\partial z} = 0$$

In a dry adiabatic, hydrostatic atmosphere the temperature $T$ must decrease with height. How quickly does the temperature decrease?

Note: Use $\theta = T \left( \frac{p_0}{p} \right)^{R/c_p}$
Lapse Rate

The adiabatic change in temperature with height is

\[
\frac{T}{\theta} \frac{\partial \theta}{\partial z} = \frac{\partial T}{\partial z} + \frac{g}{c_p}
\]

For dry adiabatic, hydrostatic atmosphere:

\[
- \frac{\partial T}{\partial z} = \frac{g}{c_p} \equiv \Gamma_d
\]

\( \Gamma_d \)  Dry adiabatic lapse rate (approx. 9.8 K/km)

In adiabatic motion, with no external source of heating, if a parcel moves up or down its temperature will change. Adiabatic does not mean isothermal. It means there is no external heating or cooling.
Lapse Rate

Temperature $T < T_0$

What is the lapse rate experienced by a rising air parcel here?

What is the lapse rate experienced by a sinking air parcel here?

Dry air $T_0$

Cooling

Warming

Paul Ullrich  Dynamics Review  Spring 2020
If dry adiabatic dynamics governed the flow, the decay rate of temperature with height should be very close to the adiabatic lapse rate in a dry atmosphere.
Lapse Rate

If dry adiabatic dynamics governed the flow, the decay rate of temperature with height should be very close to the adiabatic lapse rate in a dry atmosphere.

However, the actual large-scale atmospheric Lapse rate is closer to 6.5 K/km (this is actually the value used by the International Standard Atmosphere).

Why might this be the case?

Table: The International Standard Atmosphere (through 10km)

<table>
<thead>
<tr>
<th>Altitude (km)</th>
<th>Air density (kg/m³)</th>
<th>Temperature (K)</th>
<th>Lapse Rate (K/km)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.225</td>
<td>288.15</td>
<td>6.5</td>
</tr>
<tr>
<td>1</td>
<td>1.112</td>
<td>281.65</td>
<td>6.5</td>
</tr>
<tr>
<td>2</td>
<td>1.007</td>
<td>275.15</td>
<td>6.5</td>
</tr>
<tr>
<td>3</td>
<td>0.909</td>
<td>268.65</td>
<td>6.5</td>
</tr>
<tr>
<td>4</td>
<td>0.819</td>
<td>262.15</td>
<td>6.5</td>
</tr>
<tr>
<td>5</td>
<td>0.736</td>
<td>255.65</td>
<td>6.5</td>
</tr>
<tr>
<td>6</td>
<td>0.660</td>
<td>249.15</td>
<td>6.5</td>
</tr>
<tr>
<td>7</td>
<td>0.590</td>
<td>242.75</td>
<td>6.5</td>
</tr>
<tr>
<td>8</td>
<td>0.526</td>
<td>236.25</td>
<td>6.5</td>
</tr>
<tr>
<td>9</td>
<td>0.467</td>
<td>229.75</td>
<td>6.5</td>
</tr>
<tr>
<td>10</td>
<td>0.413</td>
<td>223.25</td>
<td>6.5</td>
</tr>
</tbody>
</table>
The atmosphere is not dry – motion is not dry adiabatic.

If air reaches saturation (and the conditions are right for cloud formation), vapor will condense to liquid or solid and release energy.

So, in general, we have $J \neq 0$

**Definition:** The saturated adiabatic lapse rate (SALR) is the rate at which air cools with height when the air parcel is fully saturated. This is approximately 5 K/km, but dependent on the air temperature.

But since the atmosphere not completely saturated everywhere, the actual lapse rate is closer to 6.5 K/km.
Lapse Rate

What is the lapse rate experienced by a rising air parcel here?

What is the lapse rate experienced by a sinking air parcel here?

Saturated air

Cooling

Temperature $T < T_0$

Warm, dry foehn winds

Temperature $T > T_0$
**Definition**: A fohn or foehn wind is a dry, warm down-slope wind that occurs along the lee (downward side) of the mountain range.
Thank You!