Part 5: Mixed Rossby-Gravity Waves
Trapped waves decay exponentially rapidly away from a boundary (such as the equator or a coastline). Trapped waves include:

- Equatorial (and coastal) Kelvin waves
- Equatorial Rossby (ER) waves
- Mixed Rossby-Gravity (MGR) waves at the equator
- Equatorial inertia-gravity waves
Equatorial Rossby and Mixed Rossby-Gravity Waves

- Shallow water equations, no topography
- Coriolis parameter at the equator is approximated by equatorial $\beta$-plane:

$$f \approx \beta y \approx \left( \frac{2\Omega}{a} \right) y$$

$$\frac{Du}{Dt} - \beta y v + \frac{\partial \Phi}{\partial x} = 0$$
$$\frac{Dv}{Dt} + \beta y u + \frac{\partial \Phi}{\partial y} = 0$$
$$\frac{D\Phi}{Dt} + \Phi \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0$$  \hspace{1cm} \text{with} \hspace{1cm} \Phi = gh$$
Equatorial Rossby and Mixed Rossby-Gravity Waves

Linearize shallow water equations about a state at rest with mean height $H$

$u = u', v = v', h = H + h'$

\[
\begin{align*}
\frac{\partial u'}{\partial t} - \beta y v' + \frac{\partial \Phi'}{\partial x} &= 0 \quad (1) \\
\frac{\partial v'}{\partial t} + \beta y u' + \frac{\partial \Phi'}{\partial y} &= 0 \quad (2) \\
\frac{\partial \Phi'}{\partial t} + gH \left( \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} \right) &= 0 \quad (3)
\end{align*}
\]

with $\Phi' = gh'$

Seek wave solutions of the form (allow amplitudes to vary in $y$):

$$(u', v', \Phi') = \left( \hat{u}(y), \hat{v}(y), \hat{\Phi}(y) \right) \exp(i(kx - \nu t))$$
Equatorial Rossby and Mixed Rossby-Gravity Waves

Yields the system

\[-i\nu\hat{u} - \beta y\hat{v} + ik\hat{\Phi} = 0 \quad (1)\]

\[-i\nu\hat{v} + \beta y\hat{u} + \frac{\partial \hat{\Phi}}{\partial y} = 0 \quad (2)\]

\[-i\nu\hat{\Phi} + gH \left( ik\hat{u} + \frac{\partial \hat{v}}{\partial y} \right) = 0 \quad (3)\]

Solve (1) for \( \hat{u} \):

\[\hat{u} = i\beta y \hat{v} + k \frac{\hat{\Phi}}{\nu} \quad (4)\]
Equatorial Rossby and Mixed Rossby-Gravity Waves

Plug (4) into (2) and (3), rearrange terms

\[
(\beta^2 y^2 - \nu^2) \hat{v} - ik \beta y \hat{\Phi} - i\nu \frac{\partial \hat{\Phi}}{\partial y} = 0 \quad (5)
\]

\[
(\nu^2 - gH k^2) \hat{\Phi} + i\nu g H \left( \frac{\partial \hat{v}}{\partial y} - \frac{k}{\nu} \beta y \hat{v} \right) = 0 \quad (6)
\]

Solve (6) for \(\hat{\Phi}\):

\[
\hat{\Phi} = -\frac{i\nu g H}{(\nu^2 - gH k^2)} \left( \frac{\partial \hat{v}}{\partial y} - \frac{k}{\nu} \beta y \hat{v} \right) \quad (7)
\]
Plug (7) into (5) and rearrange terms. Yields second-order differential equation for amplitude function

\[ \frac{\partial^2 \hat{v}}{\partial y^2} + \left[ \left( \frac{\nu^2}{gH} - k^2 - \frac{k}{\nu} \beta \right) - \frac{\beta^2 y^2}{gH} \right] \hat{v} = 0 \]  

More complicated equation than before since the coefficient in square brackets is not constant (depends on y)

Before discussing the solution in detail, let's first look at two asymptotic limits when either \( H \to \infty \) or \( \beta = 0 \)
Equatorial Rossby and Mixed Rossby-Gravity Waves

Asymptotic limit $H \to \infty$

Means that the motion is non-divergent, consequence of the continuity equation (3):

$$\left( \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} \right) = -\frac{1}{gH} \frac{\partial \Phi'}{\partial t} \to 0 \quad \text{for} \quad H \to \infty$$

Equation (8) becomes

$$\frac{\partial^2 \hat{v}}{\partial y^2} + \left[ -k^2 - \frac{k}{\nu} \beta \right] \hat{v} = 0 \quad (9)$$
Equatorial Rossby and Mixed Rossby-Gravity Waves

Asymptotic limit $H \to \infty$

$$\frac{\partial^2 \hat{v}}{\partial y^2} + \left[ -k^2 - \frac{k}{\nu} \beta \right] \hat{v} = 0 \quad (9)$$

Solutions for the amplitude function $\hat{v}$ in (9) exists of the form:

$$\hat{v} = v_0 \exp(i\ell y)$$

Provided that the frequency $\nu$ satisfies the Rossby wave dispersion relationship:

$$\nu = \frac{-\beta k}{k^2 + \ell^2}$$

It shows that for non-divergent barotropic flow, equatorial dynamics are in no way special Earth’s rotation enters in form of $\beta$, not $f$: Rossby wave
Equatorial Rossby and Mixed Rossby-Gravity Waves

Asymptotic limit $\beta=0$

All influence of rotation is eliminated. Due to linearity, solutions of (9) exists of the form $\hat{v} = v_0 \exp(ily)$

Equation (8) reduces to the shallow water gravity model

$$\frac{\partial^2 \hat{v}}{\partial y^2} + \left[ \frac{\nu^2}{gH} - k^2 \right] \hat{v} = 0$$

Non-trivial solutions if frequency $\nu$ satisfies

$$\nu = \pm \sqrt{gH (k^2 + \ell^2)}$$

Pure gravity wave response (eastward and westward)
Equatorial Rossby and Mixed Rossby-Gravity Waves

In the general case with

\[
\frac{\partial^2 \hat{v}}{\partial y^2} + \left[ \left( \frac{\nu^2}{gH} - k^2 - \frac{k}{\nu} \beta \right) - \frac{\beta^2 y^2}{gH} \right] \hat{v} = 0 \quad (8)
\]

we seek solutions for the meridional distribution of \( \hat{v} \) with boundary condition

\[
\hat{v} \rightarrow 0 \quad \text{for} \quad |y| \rightarrow \infty
\]

For small \( y \) the coefficient in square brackets is positive and solutions oscillate in \( y \). For large \( y \) the coefficient is negative, solutions either grow or decay.

Only the **decaying** solution satisfies boundary condition.
Equatorial Rossby and Mixed Rossby-Gravity Waves

Matsuno (1966) showed that decaying solutions only exist if the constant part of the coefficient in square brackets satisfies the relationship

$$\frac{\sqrt{gH}}{\beta} \left( \frac{\nu^2}{gH} - k^2 - \frac{k}{\nu} \beta \right) = 2n + 1$$

$$n = 0, 1, 2, \ldots$$

This is a cubic dispersion relation for frequency $\nu$.

It determines the allowed frequencies $\nu$ of equatorially trapped waves for the zonal wavenumber $k$ and meridional mode number $n$. 
Equatorial Rossby and Mixed Rossby-Gravity Waves

In (8) replace $y$ by the nondimensional meridional coordinate

$$\xi = \left[ \left( \frac{\beta}{\sqrt{gH}} \right)^{1/2} y \right]$$

Then the solution has the form

$$\hat{v}(\xi) = v_0 H_n(\xi) \exp \left( -\frac{\xi^2}{2} \right)$$

Where $v_0$ is a constant and $H_n(\xi)$ designates the $n^{th}$ Hermite polynomial. The first few Hermite polynomials have the form

$$H_0(\xi) = 1, \quad H_1(\xi) = 2\xi, \quad H_2(\xi) = 4\xi^2 - 2$$

Index $n$ corresponds to the number of N-S roots in $\nu$. 
Hermite Polynomials

Meridional (N-S) structure of the amplitude function:

\[ \hat{v}(\xi) = v_0 H_n(\xi) \exp\left(\frac{-\xi^2}{2}\right) \quad n = 0, 1, 2, \ldots \]

Hermite polynomials:

\[ H_n(\xi) = (-1)^n \exp(\xi^2) \left[ \frac{d^n}{d\xi^n} \left( \exp\left(\xi^{-2}\right) \right) \right] \]
Equatorial Rossby and Mixed Rossby-Gravity Waves

In general, the three $\nu$ solutions of

$$\frac{\sqrt{gH}}{\beta} \left( \frac{\nu^2}{gH} - k^2 - \frac{k}{\nu \beta} \right) = 2n + 1 \quad n = 0, 1, 2, \ldots$$

can be interpreted as an equatorially trapped

1) Eastward moving inertia-gravity wave
2) Westward moving inertia-gravity wave
3) Westward moving equatorial Rossby wave

However, the case $n = 0$ must be treated separately: For $n = 0$ the meridional velocity perturbation has a Gaussian distribution centered at the equator
Equatorial Rossby and Mixed Rossby-Gravity Waves

For $n = 0$ (special case) the dispersion relation factors as

$$\left( \frac{\nu}{\sqrt{gH}} - k - \frac{\beta}{\nu} \right) \left( \frac{\nu}{\sqrt{gH}} + k \right) = 0$$

The root $\nu = -k \sqrt{gH}$ (westward propagating gravity wave) is not permitted, violates an earlier assumptions when eliminating $\hat{\Phi}$

The two other roots are

$$\nu = k \sqrt{gH} \left[ \frac{1}{2} \pm \sqrt{1 + \frac{4\beta}{k^2 \sqrt{gH}}} \right]$$
The **positive** root corresponds to an **eastward moving** equatorial **inertia-gravity** wave.

The **negative** root corresponds to a **westward moving** wave, which resembles:

- An equatorial (westward moving) **inertia-gravity (IG)** wave for long zonal scale ($k \to 0$)
- An **equatorial Rossby (ER)** wave for zonal scales characteristic of synoptic-scale disturbances

This mode ($n=0$, westward moving) is therefore called **Mixed Rossby-Gravity (MRG)** wave.
Equatorial Mixed Rossby-Gravity Waves

Plane view of horizontal velocity and height perturbations in the \((n=0)\) Mixed Rossby-Gravity (MRG) wave, propagates westward.