A Rotational View of the Atmosphere

Chapter 4

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Part 1: Vorticity
Question: What are the goals of dynamic meteorology?

1. Understand the structure of atmospheric motions (diagnosis)

2. Predict future atmospheric motions (prognosis)

We will now focus on prognosis
Wind Around a System

At the simplest level, the atmosphere can be described using vortex dynamics.

Definition: A vortex is an area of closed, circular or near-circular fluid motion.

Definition: Vorticity is a measure of the local spinning motion of the flow. It consists of a vector which denotes the local axis of rotation and the local magnitude (or rotation rate).
Rotation in the Flow

In accordance with the Helmholtz theorem, 2D fluid motion can be thought of as being composed of a **rotational component** and a **divergent component**.

Rotation is important in the development of high and low pressure systems.

There is an interplay between rotation, divergence and vertical motion.
Types of Flow

Consider an arbitrary Eulerian region embedded in a flow.
Divergent Motion

Fluid motion associated with this region can be described in terms of (a) the motion of fluid in / out of this region (flux).
Gauss’ Theorem

Flow into / out of a region must be associated with a particular region.

On the other hand, divergence is a closely connected concept which is described at a point.

These concepts are connected via Gauss’ theorem:

\[ \int u \cdot dn = \int (\nabla \cdot u) dV \]

Flow in / out of a region

Divergence of the flow field integrated over the region
Divergent Motion

The **divergent component** of the flow is used to describe flow into or away from the point.
The 3D divergence operator has the form

\[ \nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \]

The 2D divergence operator (in the horizontal plane) has the form

\[ \nabla_h \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \]

Note the subscript
Rotational Motion

Fluid motion associated with this region can be described in terms of (b) the motion of fluid around this region.
Circulation

Fluid motion associated with this region can be described in terms of (b) the motion of fluid around this region.

Definition: The circulation of a flow about some curve is the integral of the tangential velocity around that curve.

\[ C \equiv \oint u \cdot dl \]
Circulation

$C \equiv \oint u \cdot dl$

Circulation is a measure of rotational part of the flow within a region.

It is an analogue to angular momentum, and so induces a conservation law which describes its evolution. The “direction” of a circulation is defined by which direction we go around the circuit.

**Cyclonic motion**  
Counterclockwise (in the Northern hemisphere)

**Anticyclonic motion**  
Clockwise (in the Northern hemisphere)
**Stokes’ Theorem**

*Circulation* is a concept which is valid along a *closed curve*.

On the other hand, *vorticity* is a closely connected concept which is described *at a point*.

These concepts are connected via Stokes’ theorem:

\[
\oint u \cdot dl = \int (\nabla \times u) \cdot dS
\]

- Circulation about a curve
- Vorticity field over the surface bounded by the curve
The **vortical component** of the flow is used to describe rotation about a point.
In accordance with geostrophic balance, **positive vorticity** is associated with **cyclonic rotation** in the northern hemisphere (low pressure systems).

Predictions to changes in vorticity are then equivalent to predictions in changes in pressure.

The first computer forecasts only predicted the changes in vorticity and still did a decent job...

Analogously, **negative vorticity** is associated with **anti-cyclonic rotation** in the northern hemisphere (high pressure systems).
Vorticity

The 3D vorticity is associated with the curl of the vector field:

\[ \nabla \times \mathbf{u} = i \left( \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) + j \left( \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) + k \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \]

These components represent vorticity due to over-turning motions and vertical wind shear.

This component represents vorticity confined to a horizontal surface. This type of motion is the most important for understanding the dynamic evolution of large-scale weather systems.
**Vorticity**

For diagnostic purposes, the vertical component of vorticity is the most important:

\[ k \cdot (\nabla \times u) \]

Hence, the **2D vorticity** is associated with the quantity:

\[ \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \]

This is the **relative vorticity**, since it is computed locally and does not take into account the rotation of the coordinate system.
Consider only the horizontal component of the vorticity:
**Vorticity**

The **3D vorticity** is associated with the curl of the vector field:

\[
\nabla \times \mathbf{u} = i \left( \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) + j \left( \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) + k \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)
\]

Since \( \mathbf{u} \) is the velocity in the inertial coordinate system, both rotation of the planet and relative rotation need to be accounted for:

**Definition:** The component of vorticity due to relative motions is called **relative vorticity**.
**Vorticity**

Since angular momentum is only conserved in an inertial frame, rotation of the coordinate system must be accounted for.
Maximum rotation of vertical column

No rotation of vertical column
Planetary Vorticity

**Definition:** The component of vorticity due to the rotation of the Earth is called *planetary vorticity.*

\[ f = k \cdot (\nabla \times \mathbf{u}_{\text{earth}}) = 2\Omega \sin \phi \]

Planetary vorticity is the contribution to angular momentum due to the rotation of the planetary surface.
\[ f = 1.4 \times 10^{-4} \text{ s}^{-1} \]

\[ f = 1.0 \times 10^{-4} \text{ s}^{-1} \]

\[ f = 0.0 \text{ s}^{-1} \]

\[ f = 2\Omega \sin \phi \]


**Absolute Vorticity**

**Definition:** The absolute (or total) vorticity \( \eta \) is the sum of the relative vorticity and planetary vorticity.

\[
\eta \equiv \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) + f
\]

This concept arises immediately from \( \mathbf{u}_a = \mathbf{u}_{earth} + \mathbf{u} \) and linearity of the curl:

\[
\mathbf{k} \cdot (\nabla \times \mathbf{u}_a) = \mathbf{k} \cdot (\nabla \times \mathbf{u}_{earth}) + \mathbf{k} \cdot (\nabla \times \mathbf{u})
\]
Divergence

What about the contribution of the Earth’s rotation to the divergence?

$$\nabla \cdot u_{\text{earth}} = \nabla \cdot (2\Omega \sin \phi \mathbf{i})$$

$$= \frac{\partial}{\partial x} (2\Omega \sin \phi)$$

$$= \frac{1}{r \cos \phi} \frac{\partial}{\partial \lambda} (2\Omega \sin \phi)$$

$$= 0$$

The rotation of the coordinate system does not contribute to the divergence of the flow.
**Rotational vs. Divergent Flow**

Rotational Motion

\[ \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \]

\(~ 10^{-5} \text{ s}^{-1}~

Divergent Motion

\[ \delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \]

\(~ 10^{-6} \text{ s}^{-1}~

Associated 2D Diagnostic Variable

Typical Scale

Rotation is an order of magnitude more important than divergence!
At the simplest level, the atmosphere can be described using **vortex dynamics**. Vortices are dominated by rotation (curvature).

**Question:** Do all curved flows have vorticity? Are all flows with vorticity curved?

**Answer:** No. Not all curved flows have vorticity. Further, not all flows with vorticity are curved.

**Observe:** Vorticity is associated with pointwise spinning motion.
Vorticity Concepts

Vorticity is associated with pointwise spinning motion. So what happens to this stick?

\[ \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \]

\[ \delta y \sim \Delta y \]

\[ \delta x \sim \Delta x \]
Vorticity Concepts

Vorticity is associated with pointwise spinning motion. So what happens to this stick?

\[ \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \]

North

South

West

East

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Introduction to Atmospheric Dynamics
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Vorticity Concepts

Vorticity is associated with pointwise spinning motion. So what happens to this stick?

\[ \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \]

North

\[ \delta y \sim \Delta y \]

South

\[ \delta x \sim \Delta x \]

West

East

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Vorticity is associated with pointwise spinning motion. So what happens to this stick?

\[ \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \]

Pointwise rotation, but no curvature (sheared flow)
Vorticity Concepts

Relative velocity: \( \mathbf{u} = (u, v, w) \)

3D vorticity vector: \( \nabla \times \mathbf{u} = \left( \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \)

Relative vorticity: \( \zeta = \mathbf{k} \cdot (\nabla \times \mathbf{u}) = \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \)

Absolute vorticity: \( \eta = \mathbf{k} \cdot (\nabla \times \mathbf{u}_a) = \zeta + f \)

Planetary vorticity: \( f = \mathbf{k} \cdot (\nabla \times \mathbf{u}_{\text{earth}}) = 2\Omega \sin \phi \)

(local vertical component of the vorticity of the Earth)