Part 4: Vertical Motion in Pressure Coordinates
Dynamical Equations

Pressure Coordinates

Momentum Equation
\[
\frac{D\mathbf{u}_h}{Dt} + f \mathbf{k} \times \mathbf{u}_h = - \nabla_p \Phi
\]

Hydrostatic Relation
\[
\frac{\partial \Phi}{\partial p} = - \frac{R_d T}{p}
\]

Continuity Equation
\[
\left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)_p + \frac{\partial \omega}{\partial p} = 0
\]

Thermodynamic Equation
\[
\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} - S_p \omega = \frac{J}{c_p}
\]

Ideal Gas Law
\[
p = \rho R_d T
\]

Material Derivative
\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + \omega \frac{\partial}{\partial p}
\]
Vertical Motion

Connection between $w$ and $\omega$

By definition,

$$\omega \equiv \frac{Dp}{Dt} = \frac{\partial p}{\partial t} + u \frac{\partial p}{\partial x} + v \frac{\partial p}{\partial y} + w \frac{\partial p}{\partial z}$$

Hydrostatic Balance

$$\frac{\partial p}{\partial z} = -\rho g$$

Geostrophic Decomposition

$$u = u_g + u_a$$

$$\omega = \frac{\partial p}{\partial t} + (u_g + u_a) \frac{\partial p}{\partial x} + (v_g + v_a) \frac{\partial p}{\partial y} - w g \rho$$
Vertical Motion

Connection between $w$ and $\omega$

$$\omega = \frac{\partial p}{\partial t} + (u_g + u_a) \frac{\partial p}{\partial x} + (v_g + v_a) \frac{\partial p}{\partial y} - w g \rho$$

Geostrophic Balance

$$u_g = -\frac{1}{f \rho} \frac{\partial p}{\partial y}$$
$$v_g = +\frac{1}{f \rho} \frac{\partial p}{\partial x}$$

$$\omega = \frac{\partial p}{\partial t} + \left( u_a \frac{\partial p}{\partial x} + v_a \frac{\partial p}{\partial y} \right) - w g \rho$$

Question: What are the scales associated with these terms?
**Scale Analysis**

Typical scales associated with large-scale mid-latitude storm systems:

\[ U \approx 10 \text{ m s}^{-1} \]
\[ W \approx 0.01 \text{ m s}^{-1} \]
\[ L \approx 10^6 \text{ m} \]
\[ H \approx 10^4 \text{ m} \]
\[ L/U \approx 10^5 \text{ s} \]
\[ \Delta P \approx 10 \text{ hPa} = 1000 \text{ Pa} \]
\[ \rho \approx 1 \text{ kg m}^{-3} \]
\[ \Delta \rho/\rho \approx 10^{-2} \]
\[ f_0 \approx 10^{-4} \text{ s}^{-1} \]
\[ a \approx 10^7 \text{ m} \quad \text{(Radius of Earth)} \]
\[ g \approx 10 \text{ m s}^{-2} \quad \text{(Gravity)} \]
\[ \nu \approx 10^{-5} \text{ m}^2 \text{ s}^{-1} \quad \text{(Kinematic Viscosity)} \]
**Vertical Motion**

**Connection between \( w \) and \( \omega \)**

\[
\omega = \frac{\partial p}{\partial t} + \left( u_a \frac{\partial p}{\partial x} + v_a \frac{\partial p}{\partial y} \right) - wg\rho
\]

Local change in pressure:

\[
\frac{\partial p}{\partial t} \approx \frac{U \Delta P}{L} \approx 10^{-2} \text{ Pa s}^{-1}
\]

Pressure advection by ageostrophic wind:

\[
u_a \cdot \nabla_h p \approx 0.1 \times \frac{U \Delta P}{L} \approx 10^{-3} \text{ Pa s}^{-1}
\]

**Ageostrophic velocity is “small”**

Vertical velocity term:

\[
wg\rho \approx Wg\rho \approx 10^{-1} \text{ Pa s}^{-1}
\]
Vertical Motion

Connection between \( w \) and \( \omega \)

\[
\omega \approx -w g \rho
\]

From scale analysis, to a close approximation the vertical pressure velocity is given by this relationship.
**Question:** How do we diagnose the vertical pressure velocity?

Observe there are only two equations that include $\omega$:

- **Continuity Equation**

\[
\left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)_p + \frac{\partial \omega}{\partial p} = 0
\]

- **Thermodynamic Equation**

\[
\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} - S_p \omega = \frac{J}{c_p}
\]

Each of these equations leads to one diagnostic equation for $\omega$. 
**Vertical Pressure Velocity**

**Kinematic Method**

Continuity Equation

\[
\left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)_p + \frac{\partial \omega}{\partial p} = 0
\]

Geostrophic Decomposition

\[
\nabla_p \cdot (u_g + u_a) + \frac{\partial \omega}{\partial p} = 0
\]

Geostrophic Balance

\[
\frac{\partial}{\partial x} \left( -\frac{1}{f} \frac{\partial \Phi}{\partial y} \right) + \frac{\partial}{\partial y} \left( \frac{1}{f} \frac{\partial \Phi}{\partial x} \right) + \nabla_p \cdot u_a + \frac{\partial \omega}{\partial p} = 0
\]
Vertical Pressure Velocity

Kinematic Method

\[
\frac{\partial}{\partial x} \left( -\frac{1}{f} \frac{\partial \Phi}{\partial y} \right) + \frac{\partial}{\partial y} \left( \frac{1}{f} \frac{\partial \Phi}{\partial x} \right) + \nabla_p \cdot \mathbf{u}_a + \frac{\partial \omega}{\partial p} = 0
\]

Assume \( f \) is approximately constant

\[\frac{\partial \omega}{\partial p} \approx -\nabla_p \cdot \mathbf{u}_a\]

As observed for the case of height coordinates, vertical pressure velocity is connected to the divergence of the ageostrophic wind in pressure surfaces.
Using the Kinematic method, vertical pressure velocity is diagnosed from the mean layer horizontal divergence.
Vertical Pressure Velocity

Kinematic Method

\[
\omega(p_1) \approx \omega(p_2) - (p_1 - p_2) \left( \frac{\partial \langle u \rangle}{\partial x} + \frac{\partial \langle v \rangle}{\partial y} \right)
\]

However, if the flow is very close to geostrophic balance, then the divergence is small and calculating the mean layer divergence requires an accurate representation of horizontal velocities.

Therefore: Small errors in evaluating the winds \(<u>\) and \(<v>\) lead to large errors in \(\omega\). Consequently, the Kinematic method tends to be inaccurate.
**Vertical Pressure Velocity**

**Adiabatic Method**

Thermodynamic Equation

\[
\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} - S_p \omega = \frac{J}{c_p}
\]

Assume diabatic heating is small

\[
\omega = S_p^{-1} \left( \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right)
\]

Stability parameter

Horizontal advection of temperature
**Vertical Pressure Velocity**

*Adiabatic Method*

If local time tendency is negligible (steady state):

\[
\frac{\partial T}{\partial t} \approx 0 \quad \Rightarrow \quad \omega = - \left[ \frac{-u_h \cdot \nabla T}{S_p} \right]
\]

Horizontal advection of temperature

If temperature time tendency is steady, flow is adiabatic and the atmosphere is stable:

- **Warm air advection**
  - Ascending Air: \( \omega < 0, w \approx -\omega/\rho g > 0 \)

- **Cold air advection**
  - Descending Air: \( \omega > 0, w \approx -\omega/\rho g < 0 \)
Vertical Pressure Velocity

Adiabatic Method

The adiabatic method is based on temperature advection, which is dominated by the geostrophic wind (large).

Hence, this method is a reasonable way to estimate local vertical velocity when advection is strong.