1 Derivation of Coriolis Terms

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1.1 Impact of meridional velocity on zonal velocity

Zonal angular momentum per unit mass is given by $|L_x|/m = (\Omega R^2 + uR)$, where R is the distance from the axis of rotation, Ω is the rotational frequency, and u is the zonal velocity relative to the surface.

Since the angular momentum per unit mass is conserved following the fluid parcel, we must have

$$\left[\Omega R_f^2 + u_f R_f\right] - \left[\Omega R_i^2 + u_i R_i\right] = 0, \tag{1}$$

where the subscripts i and f denote the initial and final values of these quantities.

Now assume that the fluid parcel originally at latitude φ_i is displaced slightly poleward by a distance δy. Then the radius from the axis of rotation changes by $\Delta R \approx -\delta y \sin\varphi_i$. Consequently,

$$R_f = R_i + \Delta R = R_i - \delta y \sin\varphi_i. \tag{2}$$

In order for angular momentum to be conserved, the zonal velocity of the fluid parcel must also change by an amount Δu,

$$u_f = u_i + \delta u. \tag{3}$$

Substituting (2) and (3) into (1) then gives

$$\left[\Omega(R_i - \delta y \sin\varphi_i)^2 + (u_i + \delta u)(R_i - \delta y \sin\varphi_i)\right] - \left[\Omega R_i^2 + u_i R_i\right] = 0. \tag{4}$$

We now expand the terms on the left-hand side to obtain

$$-2\Omega R_i \delta y \sin\varphi_i + \Omega \delta y^2 \sin^2\varphi_i + R_i \delta u - u_i \delta y \sin\varphi_i - \delta y \delta u \sin\varphi_i = 0. \tag{5}$$

Terms that are a product of two "perturbation" quantities (namely δy and δu) are then dropped since they will be much smaller than the remaining terms, leaving

$$R_i \delta u - 2\Omega R_i \delta y \sin\varphi_i - u_i \delta y \sin\varphi_i = 0. \tag{6}$$

We now solve this equation for δu, giving

$$\delta u = 2\Omega \delta y \sin\varphi_i + \frac{u_i}{R_i} \delta y \sin\varphi_i. \tag{7}$$

We assume that this process occurs over a time δt. Dividing the equation above through by δt and taking the limit as $\delta t \to 0$ gives

$$\frac{Du}{Dt} = \left[2\Omega \sin\varphi + \frac{u}{R}\sin\varphi\right]\frac{Dy}{Dt}. \tag{8}$$

If r is the distance of the fluid parcel from the center of the Earth, note that we can then write $R = r\cos\varphi$. Now using the fact that $v = Dy/Dt$, $f = 2\Omega\sin\varphi$ gives our final form for the Coriolis and curvature terms,

$$\frac{Du}{Dt} = fv + \frac{uv}{r}\tan\varphi. \tag{9}$$

1.2 Impact of vertical velocity on zonal velocity

The derivation is similar to the previous section, except assuming a small displacement δz in the vertical direction. Then the radius from the axis of rotation changes by $\Delta R \approx +\delta z\cos\varphi_i$. Consequently,

$$R_f = R_i + \Delta R = R_i + \delta z\cos\varphi_i. \tag{10}$$

Substituting (10) and (3) into (1) then gives

$$\left[\Omega(R_i + \delta z\cos\varphi_i)^2 + (u_i + \delta u)(R_i + \delta z\cos\varphi_i)\right] - \left[\Omega R_i^2 + u_i R_i\right] = 0. \tag{11}$$

Expanding the terms on the left-hand side gives

$$2\Omega R_i\delta z\cos\varphi_i + \Omega\delta z^2\cos^2\varphi_i + R_i\delta u + u_i\delta z\cos\varphi_i + \delta z\delta u\cos\varphi_i = 0. \tag{12}$$

Terms that are a product of two perturbation quantities are then dropped,

$$R_i\delta u + 2\Omega R_i\delta z\cos\varphi_i + u_i\delta z\cos\varphi_i = 0. \tag{13}$$

This equation is solved for δu,

$$\delta u = -2\Omega\delta z\cos\varphi_i - \frac{u_i}{R_i}\delta z\cos\varphi_i. \tag{14}$$

Again dividing through by δt and taking the limit as $\delta t \to 0$,

$$\frac{Du}{Dt} = \left[-2\Omega\cos\varphi - \frac{u_i}{R_i}\cos\varphi\right]\frac{Dz}{Dt}. \tag{15}$$

Now using the fact that $w = Dz/Dt$, $f = 2\Omega\sin\varphi$ and $R = r\cos\varphi$ gives the final form for the Coriolis and curvature terms,

$$\frac{Du}{Dt} = -2\Omega w\cos\varphi - \frac{uw}{r}. \tag{16}$$

1.3 Impact of vertical velocity on meridional velocity

Meridional angular momentum per unit mass is given by $|L_y|/m = vr$, where r is the distance from the center of the Earth. Consider a small displacement in the radial direction of magnitude δr. Conservation of angular momentum then requires

$$[v_f r_f] - [v_i r_i] = 0, \tag{17}$$

where $r_f = r_i + \delta r$. Denoting the change in meridional velocity by δv (i.e., $v_f = v_i + \delta v$) then leads to

$$[(v_i + \delta v)(r_i + \delta r)] - [v_i r_i] = 0, \tag{18}$$

which can be expanded as

$$v_i \delta r + r_i \delta v + \delta v \delta r = 0. \tag{19}$$

Now neglecting the product of perturbations gives

$$\delta v = -\frac{v_i \delta r}{r_i}. \tag{20}$$

Dividing through by δt and taking the limit as $\delta t \to 0$ gives

$$\frac{Dv}{Dt} = -\frac{v}{r}\frac{Dr}{Dt}. \tag{21}$$

Then using the fact that $w = Dr/Dt$ yields the expression

$$\frac{Dv}{Dt} = -\frac{vw}{r}. \tag{22}$$

This then represents the apparent force induced on the zonal velocity by the meridional force due to curvature.

1.4 Zonal velocity contributions

The impacts of rotation on the meridional component of the velocity can be obtained as follows. A fluid parcel that is initially at rest relative to the surface will be experience a centrifugal force equal to $\Omega^2 \mathbf{R}$, where $\mathbf{R}$ is the vector that connects the axis of rotation at the same latitude with the fluid parcel. If the fluid parcel is then impulsively given a zonal velocity u relative to the surface, it will experience an excessive force given by

$$\left(\Omega + \frac{u}{R}\right)^2 \mathbf{R} - \Omega^2 \mathbf{R} = \frac{2\Omega u \mathbf{R}}{R} + \frac{u^2 \mathbf{R}}{R^2}. \tag{23}$$

Then using $R = r\cos\varphi$ and $\mathbf{R} = r\cos\varphi(-\mathbf{j}\sin\varphi + \mathbf{k}\cos\varphi)$,

$$\frac{2\Omega u \mathbf{R}}{R} + \frac{u^2 \mathbf{R}}{R^2} = \left[\frac{2\Omega u r\cos\varphi\sin\varphi}{r\cos\varphi} + \frac{u^2 r\cos\varphi\sin\varphi}{r^2\cos^2\varphi}\right]\mathbf{j} + \left[\frac{2\Omega u r\cos^2\varphi}{r\cos\varphi} + \frac{u^2 r\cos^2\varphi}{r^2\cos^2\varphi}\right]\mathbf{k} \tag{24}$$

$$= \left[-2\Omega u\sin\varphi - \frac{u^2}{r}\tan\varphi\right]\mathbf{j} + \left[2\Omega u\cos\varphi + \frac{u^2}{r}\right]\mathbf{k} \tag{25}$$