ATM 265, Spring 2019
Lecture 6
Diffusion, Filters and Fixers
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Paul A. Ullrich (HH 251)
paullrich@ucdavis.edu

Slides are based on Christiane Jablonowski's talk on filtering and diffusion from the DCMIP workshop

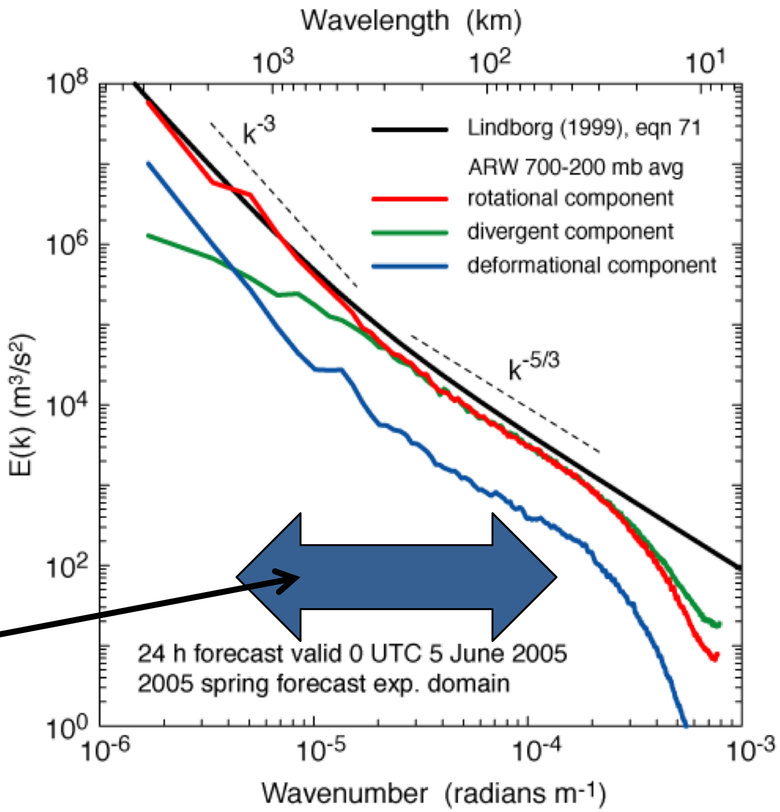
Non-Linear Equations

Non-linear differential equations, such as the ones that govern atmospheric motions include products of state variables with themselves:

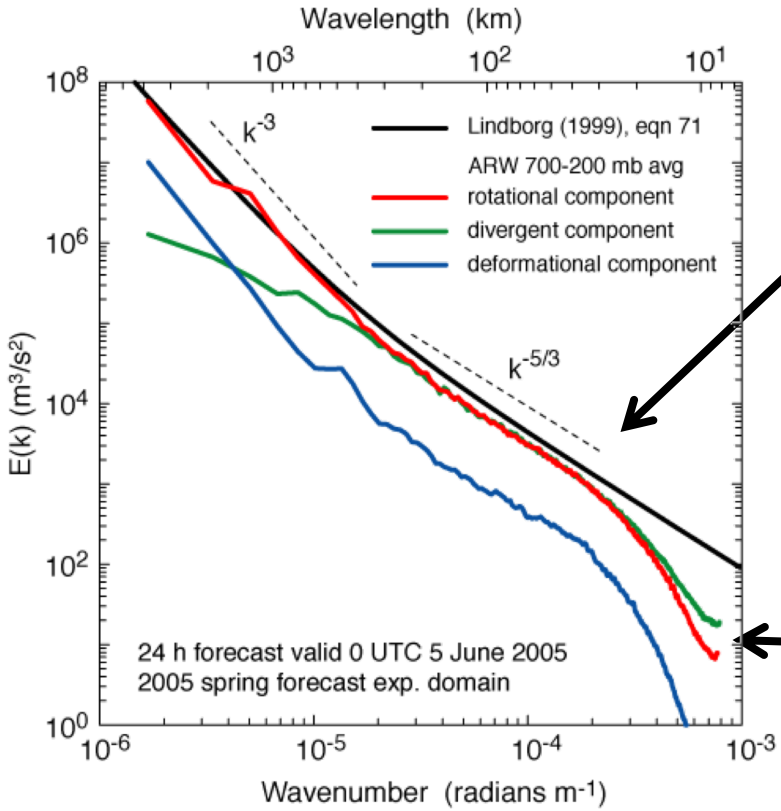
$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \mathbf{g}$$

Non-linearity!

… causes mixing between wavenumbers!



Energy Spectrum



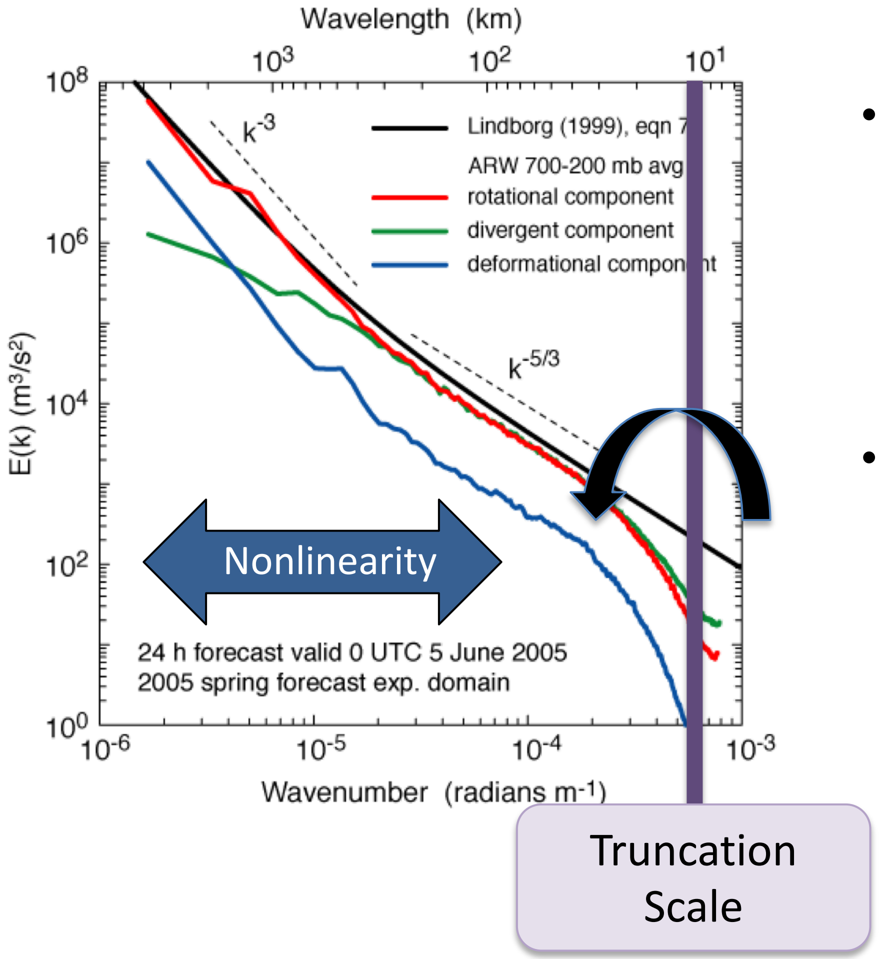
Decay of energy with wavenumber

Upward tick of energy spectrum implies weak accumulation of energy at smallest scales

Wavenumber is proportional to the inverse wavelength. Hence, larger wavenumbers = shorter waves.

Source: WRF Decomposed Spectra Spring Experiment 2005 Forecast. Courtesy of Bill Skamarock.

Energy Spectrum



- **Q:** What happens when there is a limit to the spatial scale that prevents the exchange of information between scales?

- **A:** Aliasing! If energy is conserved, it is returned to resolved scales. This can lead to an upward tick in the energy spectrum at the shortest scales.

Motivation

- All dynamical cores need some form of dissipation, either explicitly added or implicitly included via the choice of the numerical scheme.

- Due to truncation of the spatial scales:

    - Dissipation is needed to prevent an accumulation of energy at the smallest grid scales.

- Dissipation mechanisms…

  ... are often hidden in dynamical cores

  … are rarely fully documented in publications (maybe in technical reports)

  … and their coefficients are often empirically determined and resolution-dependent (*tuned*) with no physical basis

Keywords

- **Explicit diffusion (dissipation)**
  - Horizontal diffusion / hyper-diffusion
  - Divergence damping
  - Vorticity damping
  - External mode damping
  - Rayleigh friction / model top sponge layers

- **Implicit diffusion (dissipation)**
  - Order of accuracy
  - Off-centering
  - Monotonicity constraints and flux limiters

- **Filters**
  - Spectral Fast-Fourier Transform (FFT) filters
  - Digital filters (e.g. Shapiro filters)
  - Time filters (e.g. Asselin filter)

- **A posteriori Fixers:** Mass, tracer mass, total energy

Evolution Equations

Time tendency from the dynamical core (adiabatic)

Time tendency from dissipative mechanisms and fixers

$$\frac{\partial \psi}{\partial t} = Dyn(\psi) + Phys(\psi) + F(\psi)$$

Time tendency of forecast variable

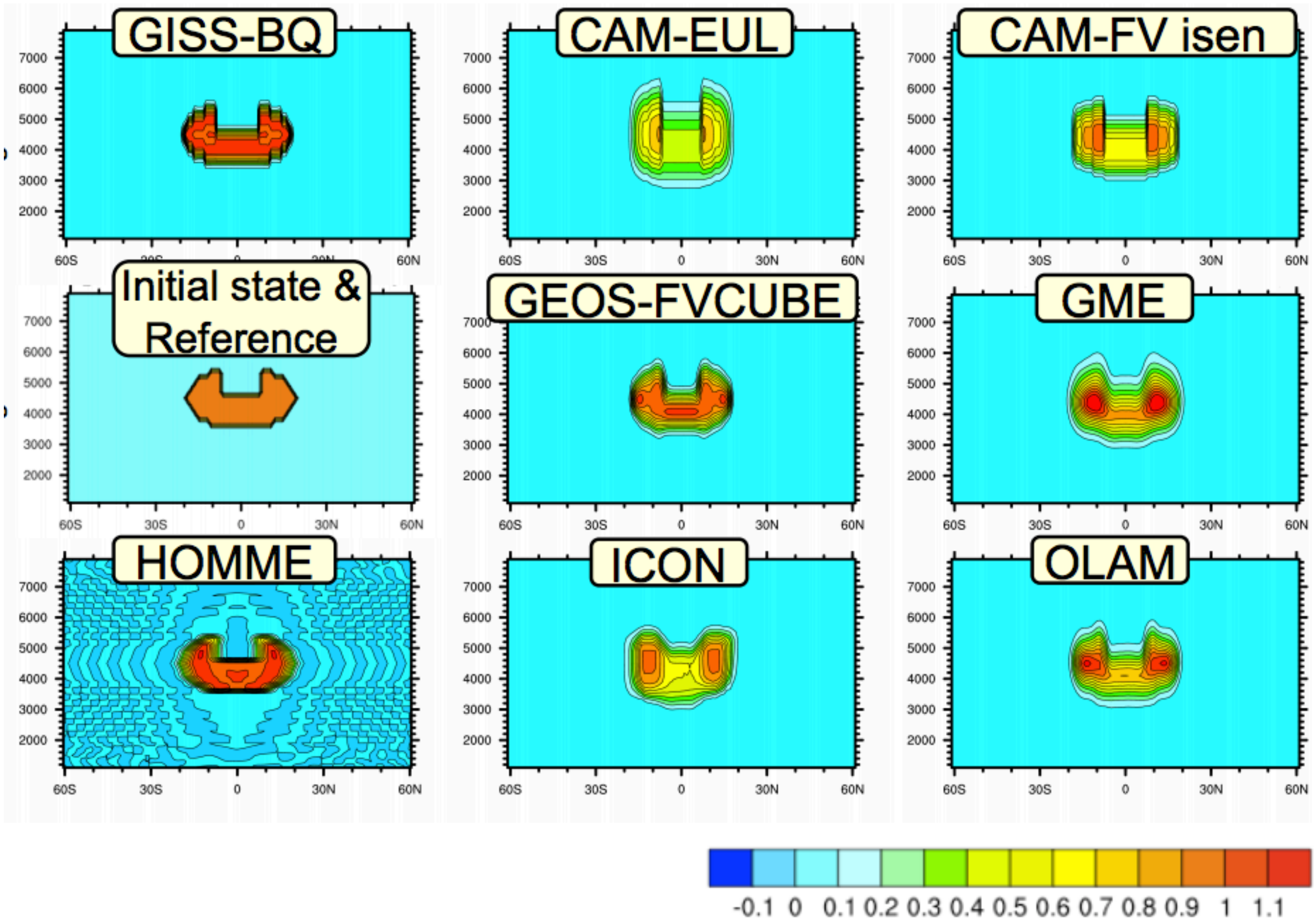
Time tendency from the physical parameterizations (diabatic)

This term is mostly considered part of the dynamical core.

Evolution Equations

Results from 8 dynamical cores during the 2008 NCAR Dynamical Core Model Intercomparison Workshop (DCMIP)

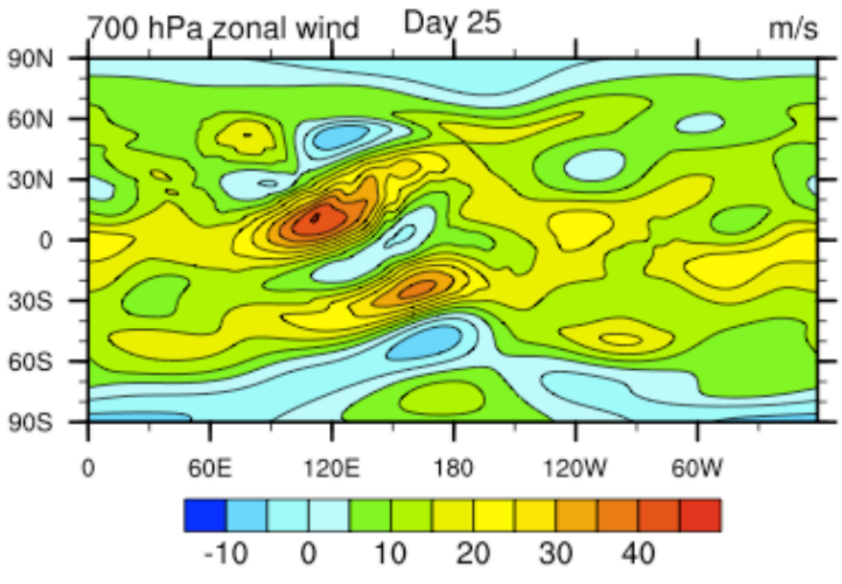
3D advection of a slotted cylinder along the equator at approximately 1° resolution



Dissipative Signatures

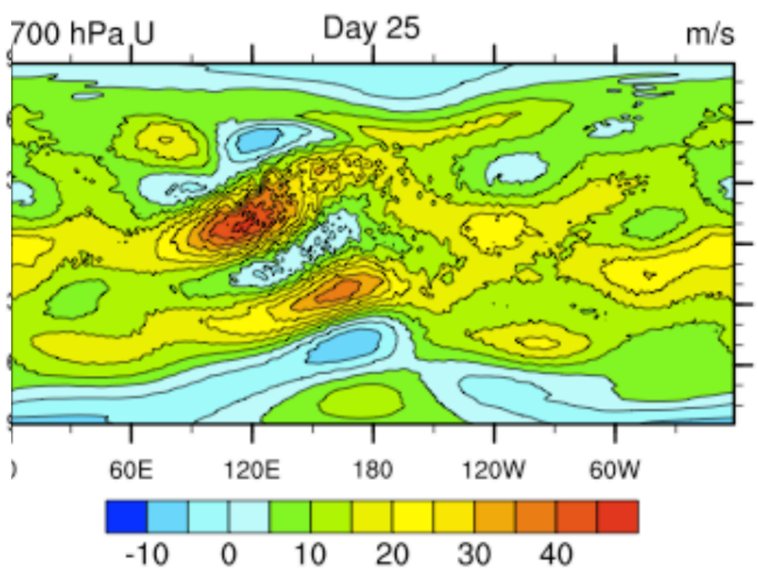
Comparison of the 700 hPa zonal wind at day 25 in CAM FV and CAM EUL with mountain wave test.

CAM FV 1° x 1° L26



With monotonicity constraint, divergence damping

CAM EUL T106L26



With horizontal 4th order hyperdiffusion with default coefficient.

Explicit Horizontal Dissipation

Diffusion applied to the prognostic variables.

- Includes regular diffusion (∇^2)
- Also hyper-diffusion $(\nabla^4, \nabla^6, \nabla^8, \ldots)$

Example: Temperature diffusion with order $2q$

$$\frac{\partial T}{\partial t} = \cdots + (-1)^{q+1} K_{2q} \nabla^{2q} T$$

Diffusion coefficent (here constant), needs to depend on horizontal resolution

Diffusion Scale Dependency

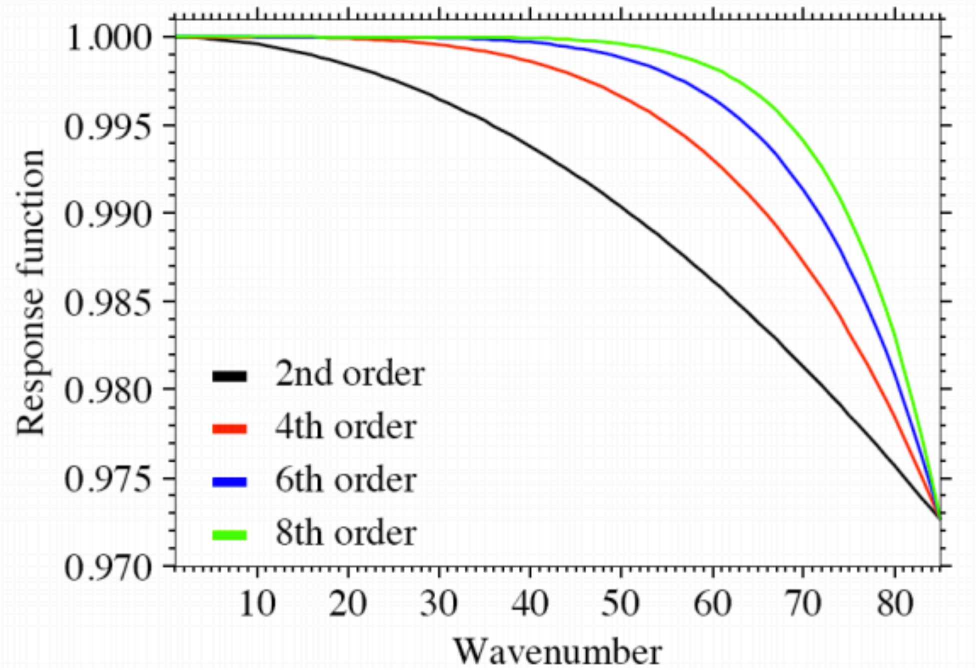
The diffusion coefficient is guided by the **e-folding time τ**: How quickly are the shortest waves damped so that the amplitude decreases by a factor of e.

Spectral models:

$$K_{2q} = \frac{1}{\tau}\left(\frac{a^2}{n_0(n_0+1)}\right)^q$$

Grid point models:

$$K_{2q} = \frac{1}{2\tau}\left(\frac{\Delta x}{2}\right)^{2q}$$

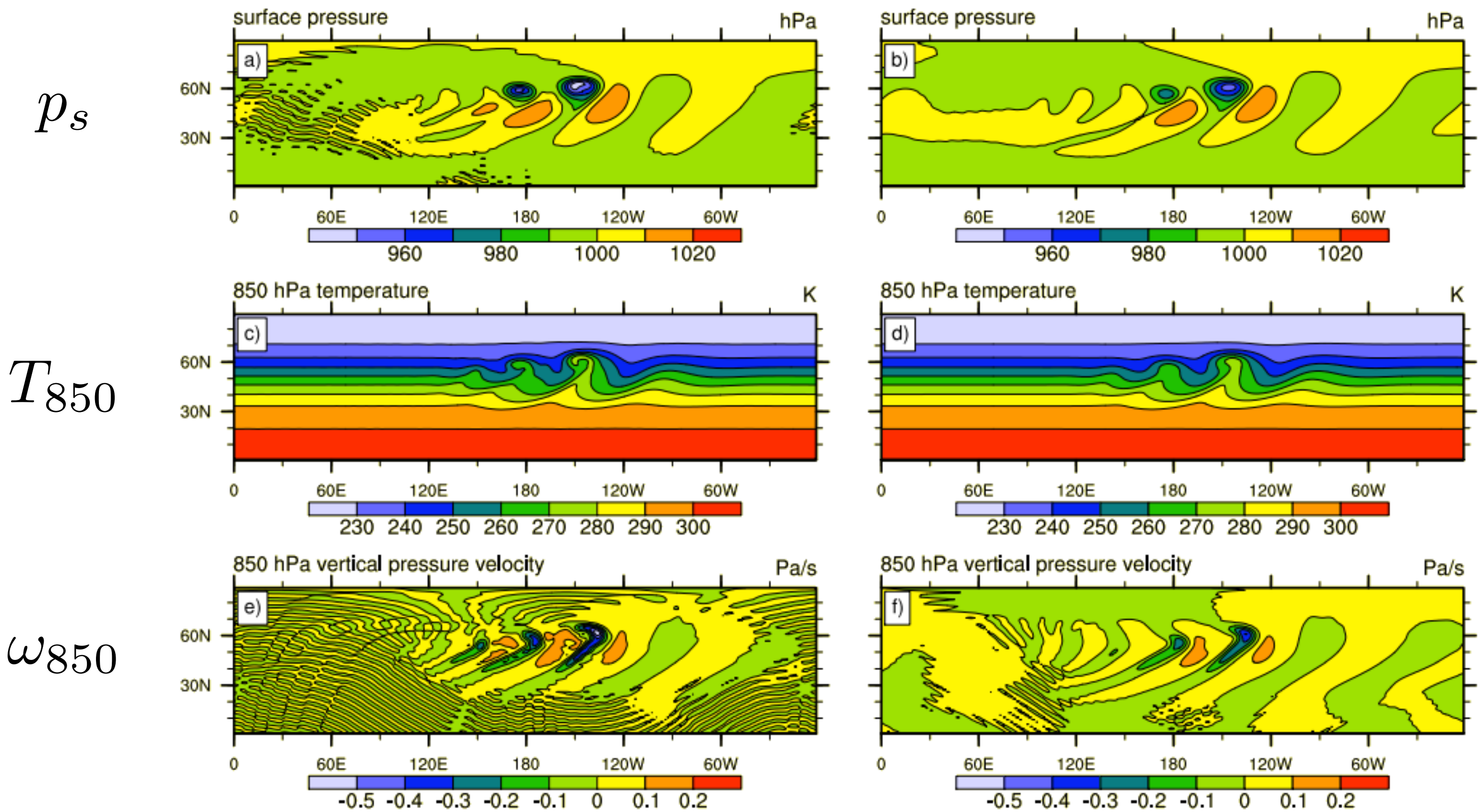


Higher order hyperdiffusion is more scale-selective, less damping at large scales (low wavenumbers)

Impact on Baroclinic Wave

CAM EUL T85L26 with two diffusion coefficients

Default $K_4 = 10^{15}$ m^4/s (8.6h) Increased $K_4 = 10^{16}$ m^4/s (0.86h)



Divergence Damping

Selectively damps the divergent part of the flow (this component is often artificially enhanced by numerical methods).

Laplacian

$$\mathbf{F}_{\mathbf{v}} = (-1)^{q+1} \nabla \left(\nu_{2q} \nabla^{2q-2} \nabla \cdot \mathbf{v} \right)$$

Gradient

Divergence

$$\frac{\partial \mathbf{v}}{\partial t} = \cdots + \nabla (c \nabla \cdot \mathbf{v})$$

Example: 2nd order div damping

$$\nabla \cdot \frac{\partial \mathbf{v}}{\partial t} = \cdots + \nabla \cdot \nabla (c \nabla \cdot \mathbf{v})$$

Apply the divergence operator

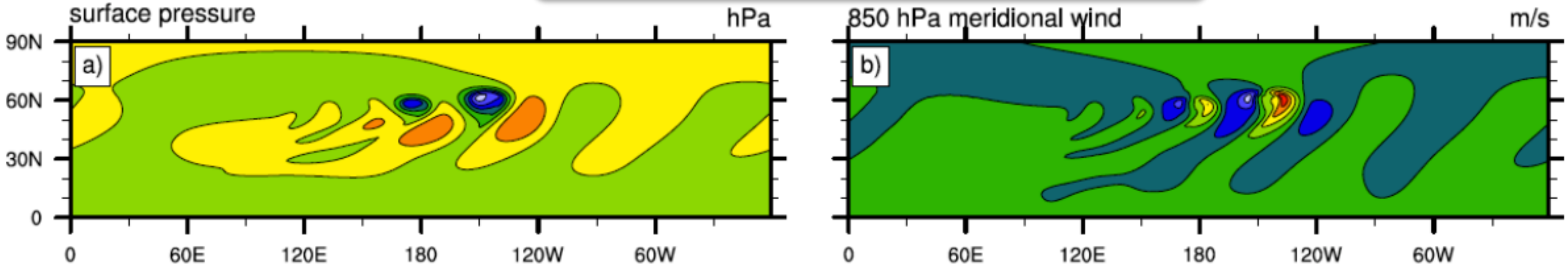
$$\frac{\partial D}{\partial t} = \cdots + \nabla^2 (c D)$$

Define $D = \nabla \cdot \mathbf{v}$

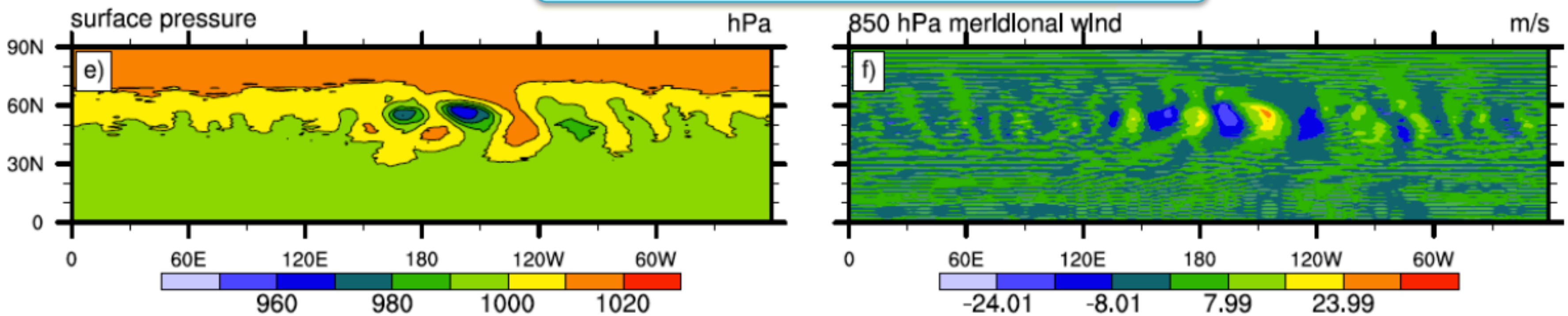
Divergence Damping

CAM FV 1° x 1° L26, baroclinic wave at day 9
(Numerical stability of CAM FV depends on divergence damping)

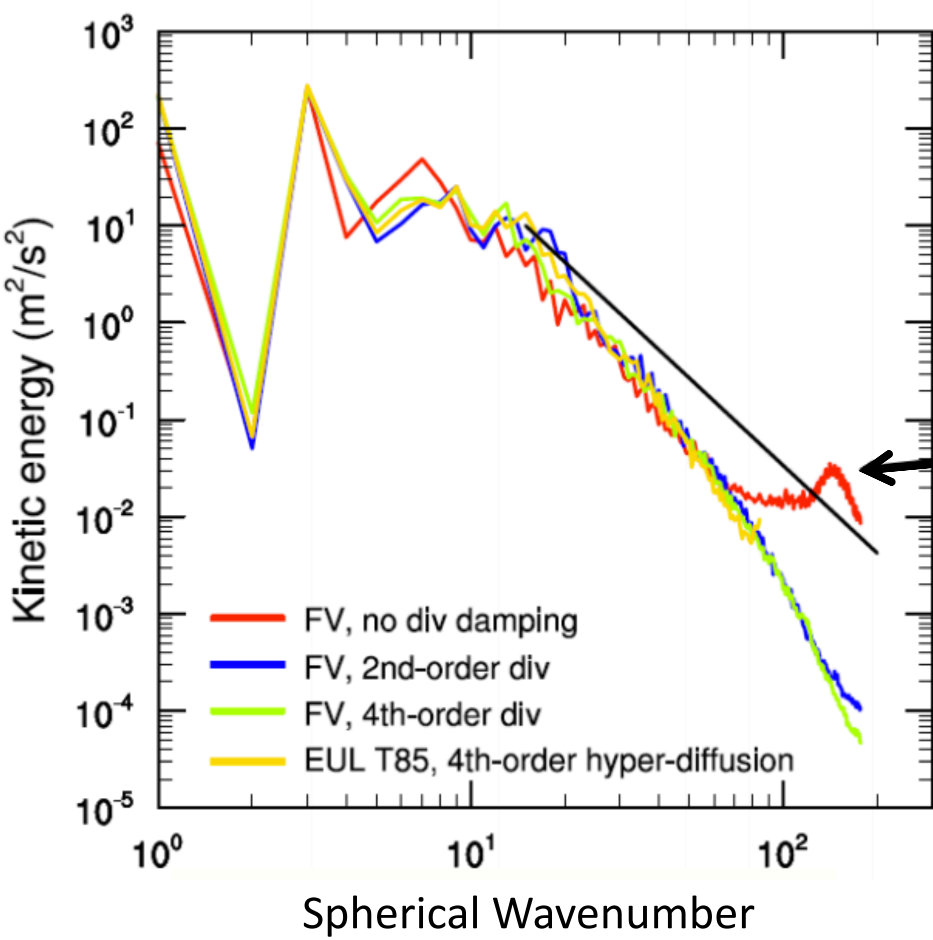
2nd order 2D divergence damping



No divergence damping



Divergence Damping



Upward tick of energy spectrum implies weak accumulation of energy at smallest scales

700 hPa KE spectra from CAM-FV and CAM-EUL baroclinic wave simulations at 1° x 1° and T85

Divergence Damping

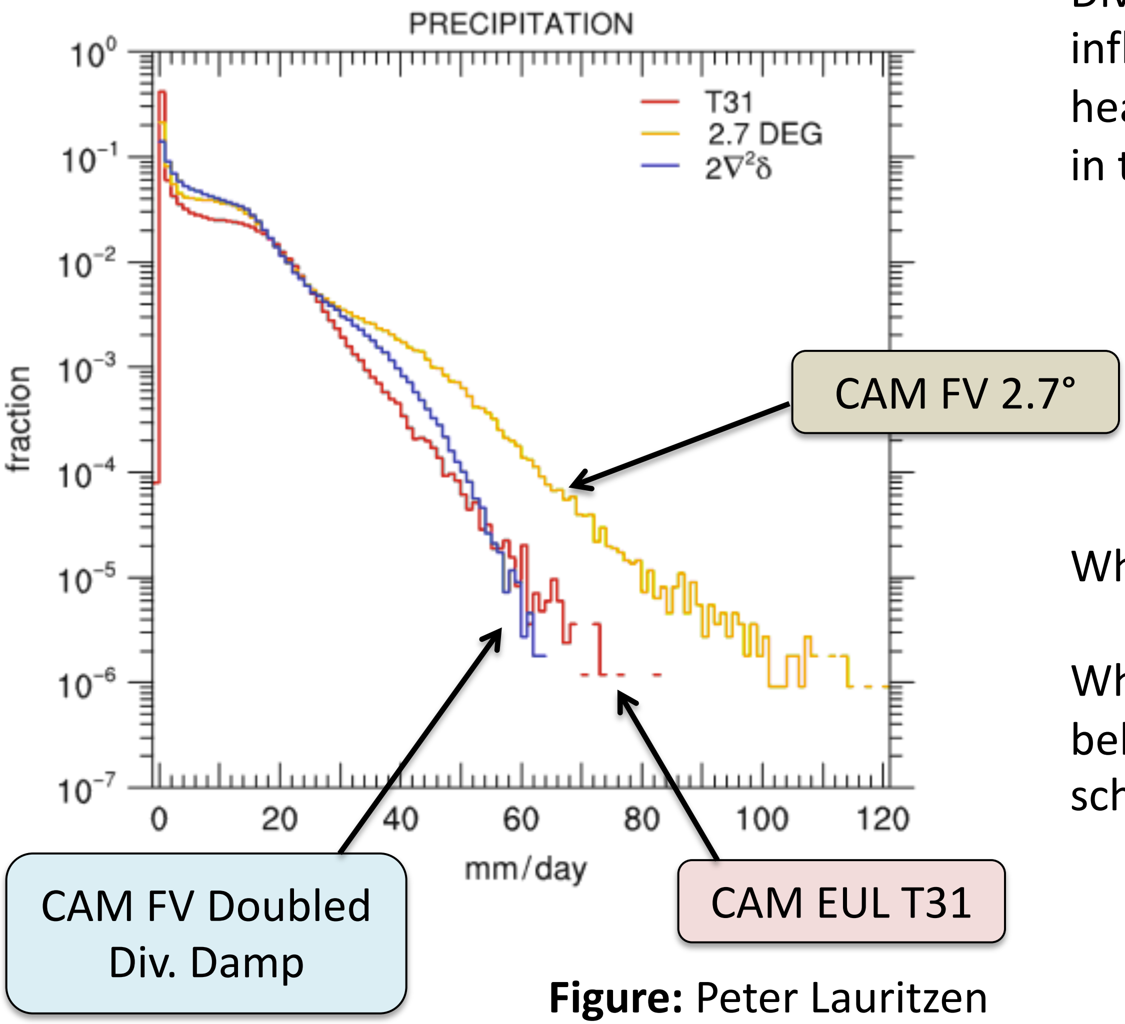


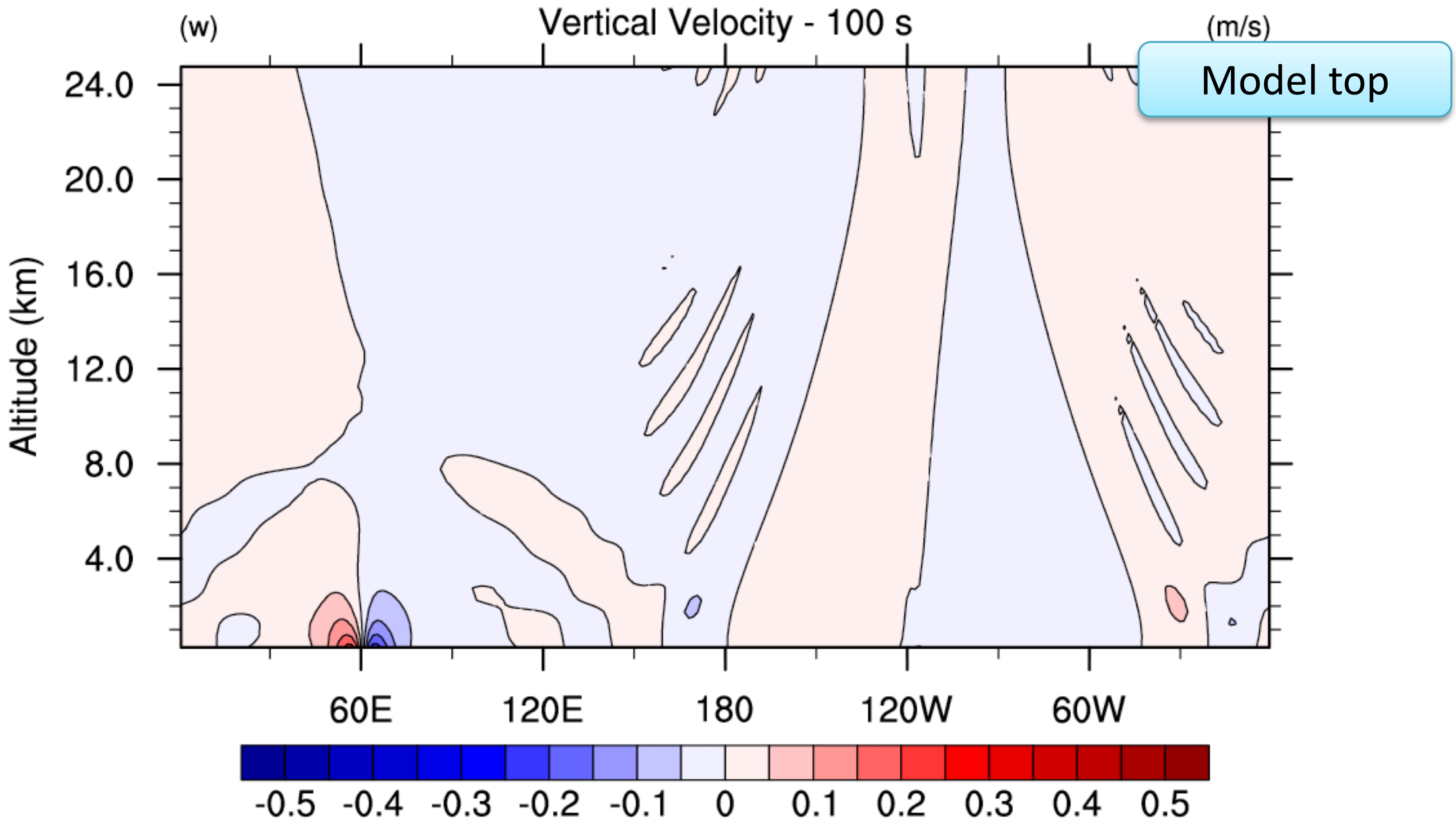
Figure: Peter Lauritzen

Divergence damping influences the likelihood of heavy precipitation events in the tropics

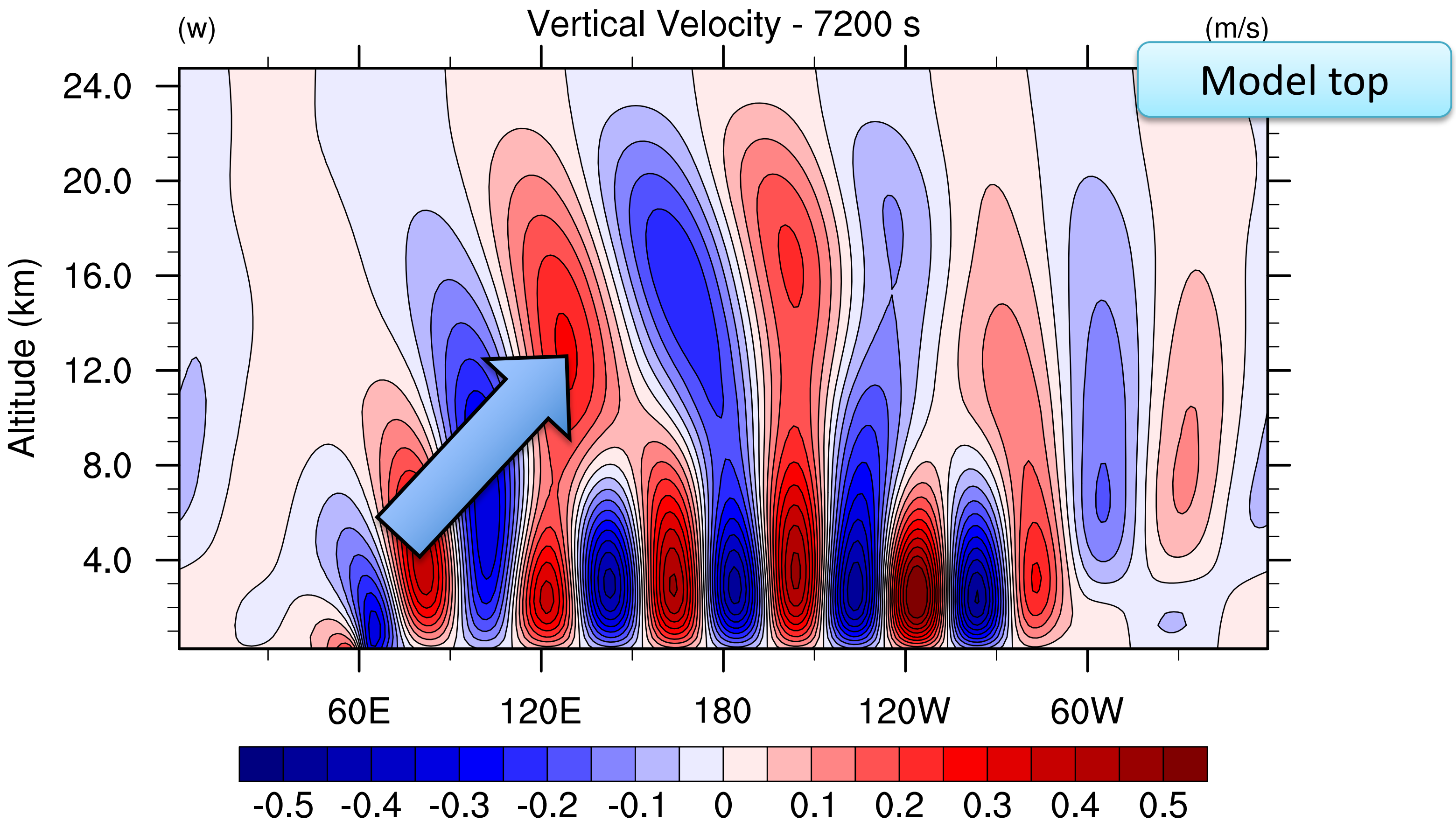
Why might this be the case?

What does it say about the behavior of the numerical scheme?

Vertically Propagating Waves



Vertically Propagating Waves



Sponge Layers / Rayleigh Friction

- A wave absorbing layer near the top of a General Circulation Model (GCM) is often desired to absorb vertically propagating waves and prevent ***wave reflection***.

- Wave reflection arises from the upper boundary condition (e.g. fixed height model top with w = 0 m/s) are perfect reflectors, undesireable.

- Practical approach: **Rayleigh Friction**

$$\frac{\partial u}{\partial t} = \cdots - \tau(u - \overline{u})$$

Damps to a background state with e-folding time τ

- Also: Height-dependent diffusion

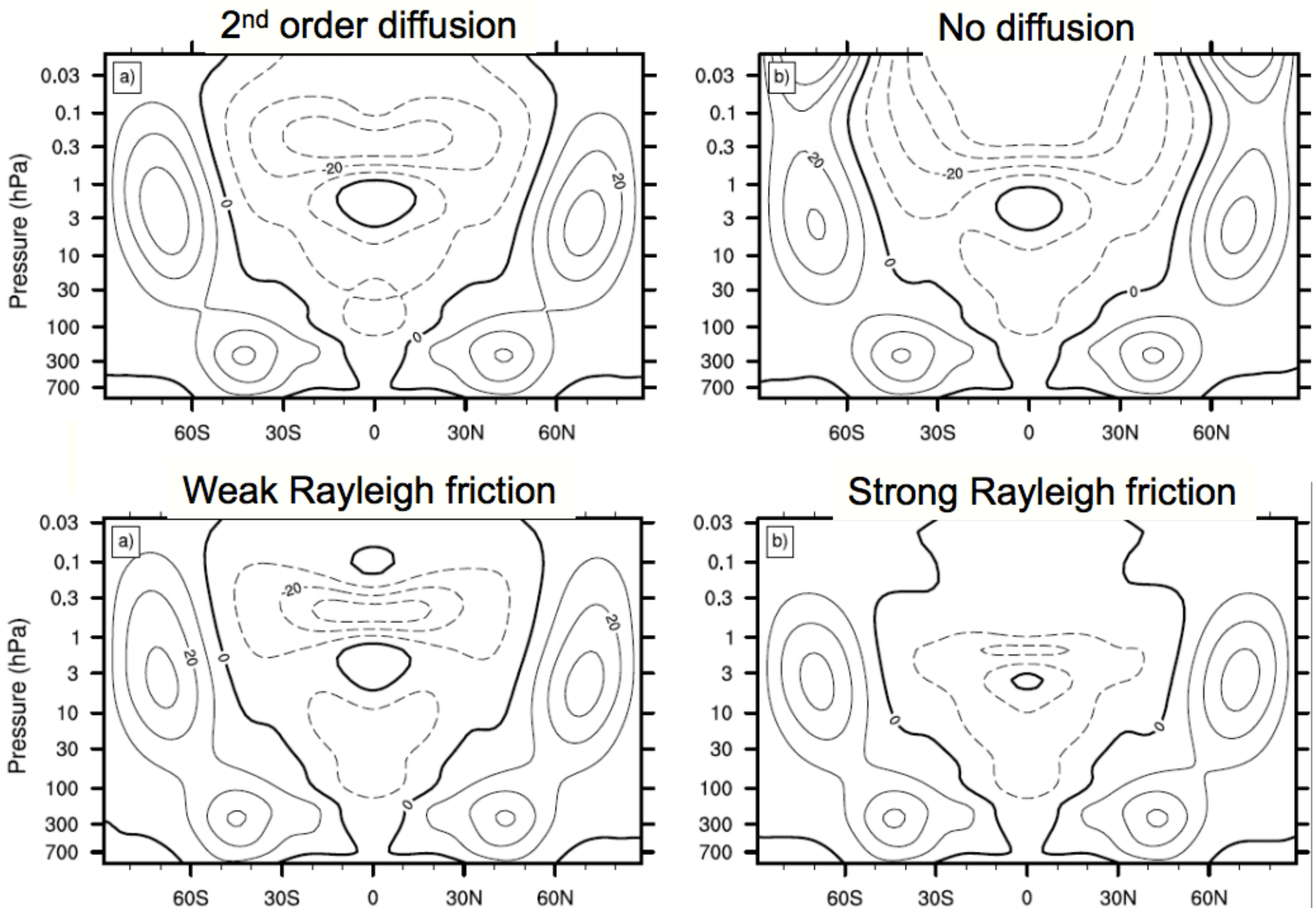
$$\frac{\partial u}{\partial t} = \cdots + K(z)\nabla^2 u$$

Compare vs. divergence damping

No physical justification

Sponge Layers / Rayleigh Friction

Held-Suarez experiment: Idealized temperature forcing, quasi-static equilibrium



Implicit Numerical Diffusion

Implicit diffusion is numerical diffusion which is inherent to a numerical scheme.

The benefit of implicit diffusion is that it typically introduces the minimum amount of diffusion necessary for maintaining stability.

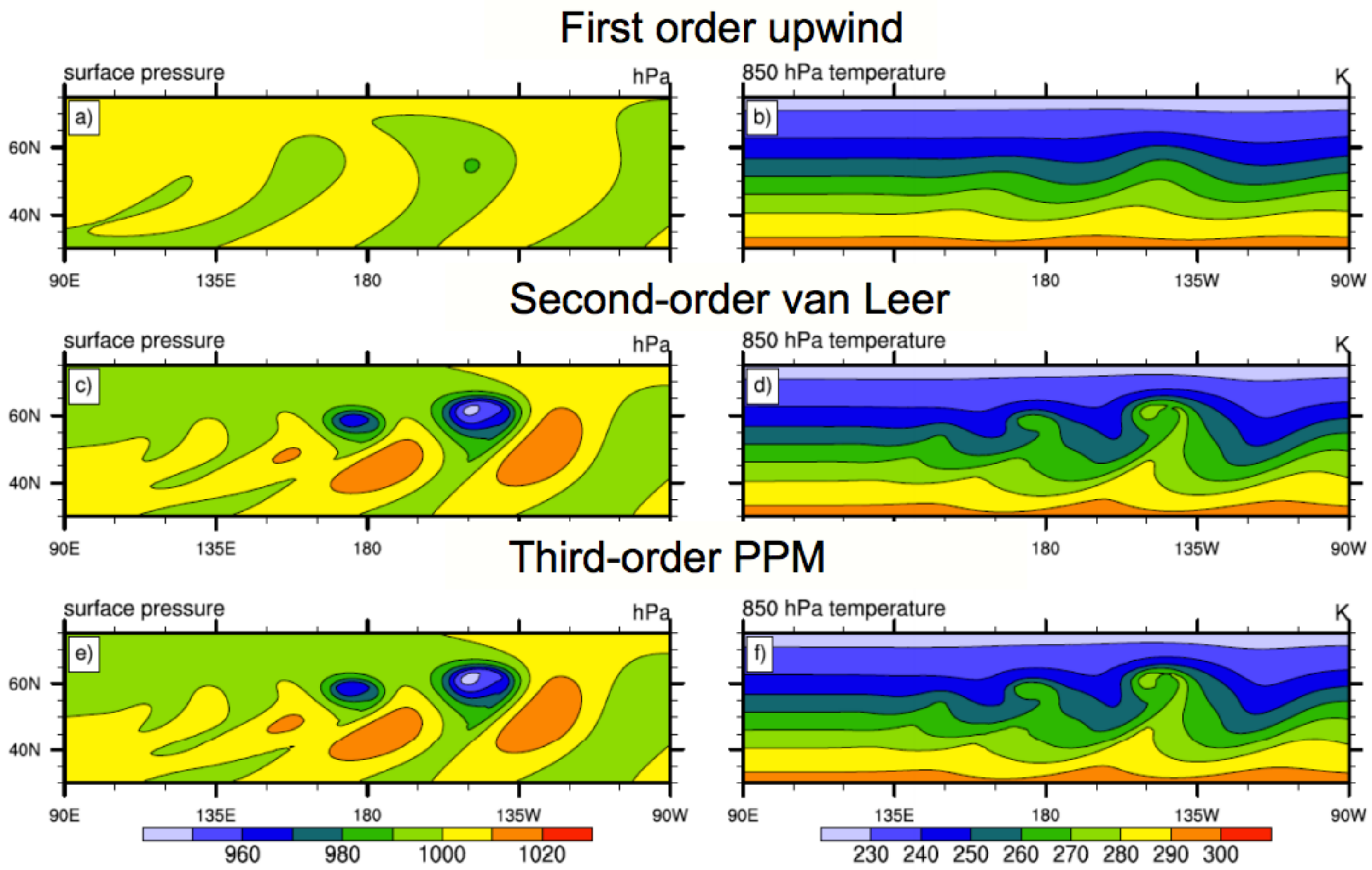
Implicit diffusion is often associated with **upwinding**; that is, taking upstream information to compute fluxes rather than centered information.

Other sources of implicit diffusion:

- Monotonicity constraints (nonlinear)
- Off-centering parameters in semi-implicit time-stepping schemes

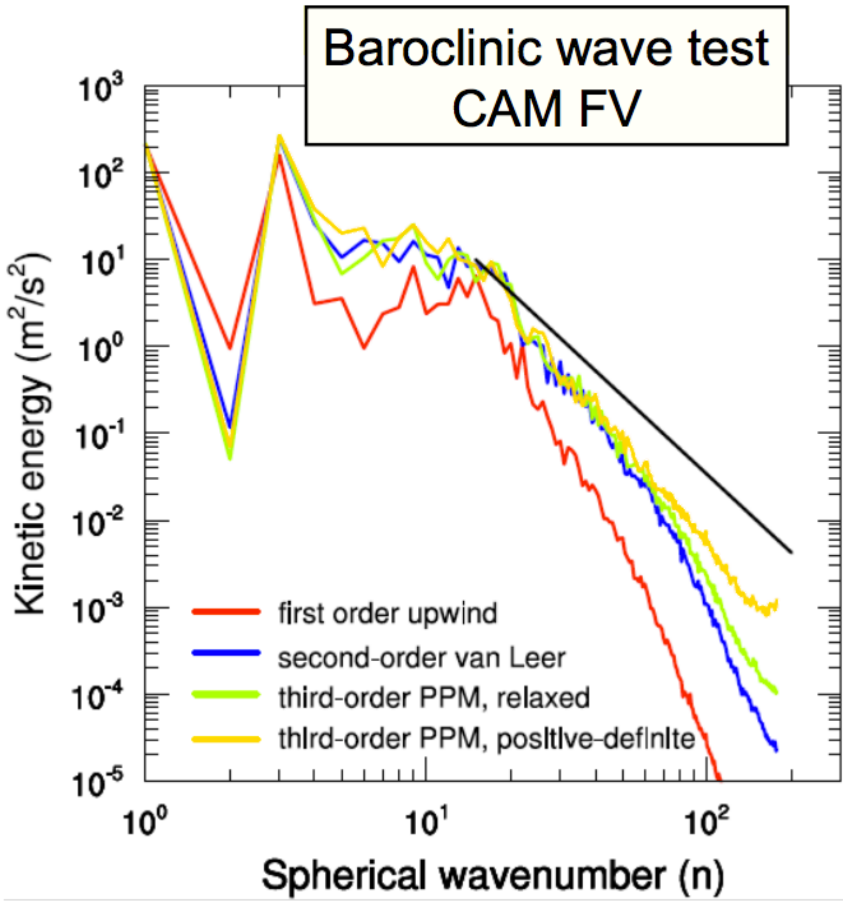
Implicit Numerical Diffusion

Change in the behavior of the baroclinic instability for three upwind-based finite-volume methods (CAM FV 1° x 1°)



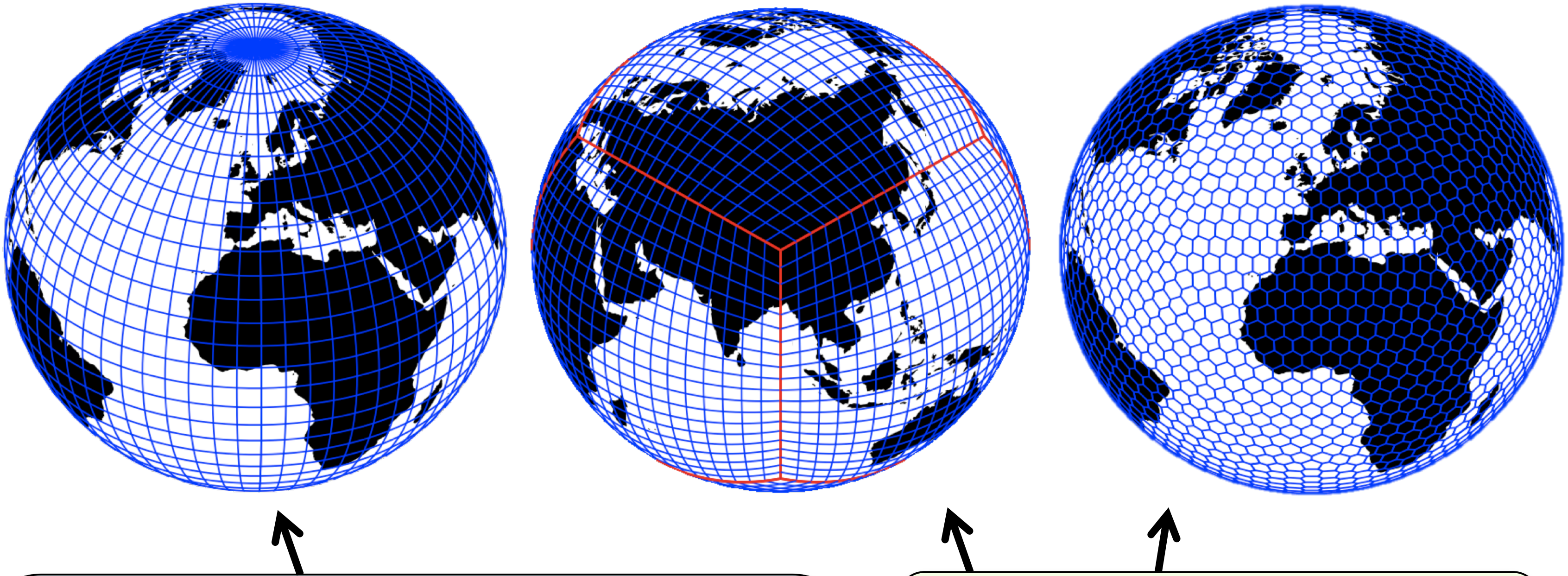
Implicit Numerical Diffusion

Change in the behavior of the baroclinic instability for three upwind-based finite-volume methods.



- 700 hPa kinetic energy spectrum (day 30) at 1° horizontal resolution
- First order upwind
- Second-order (van Leer)
- Third-order piecewise parabolic method (PPM) with relaxing
- Third-order PPM with positive-definite filter (undamped)

Spatial Filtering



Latitude-Longitude grid: On this grid the CFL condition leads to numerical instability near the poles. Polar filtering can be used to remove unstable wavelengths.

No polar filtering needed on these grids.

Spatial Filtering

The most popular filter is the 1D Fourier filter (spectral filter) used in the zonal direction (this is the direction of shortest grid spacing on the latitude-longitude grid)

Basic idea:

- Transform the grid point data into spectral space via Fourier transformation.
- Eliminate or damp high wave numbers (noise) by either setting the spectral coefficients to 0 or multiplying them with a damping coefficient
- Transform the field back into grid point space: Result is a filtered data set

Filter strength is determined by the spectral damping coefficients (can be a function of wave number), can be made to vary by scale and by latitude (often only applied above 45N)

Drawback: Needs all data along latitude ring (poor scaling on parallel computers)

Time Filtering

Used in models with 3-time level schemes (e.g. Leapfrog). Transforms temporal discretization into a temporal discretization with more desirable properties (more stability).

Most used: Robert-Asselin filter (Asselin, 1972)

Basic idea: Second-order diffusion in time

$$\frac{\partial \psi}{\partial t} = F(\psi)$$

Basic Leapfrog Scheme

$$\psi^{n+1} = \psi^{n-1} + 2\Delta t F(\psi^n)$$

Leapfrog Scheme with Robert-Asselin Filter

$$\psi^{n+1} = \overline{\psi}^{n-1} + 2\Delta t F(\psi^n)$$

$$\overline{\psi}^{n-1} = \psi^{n-1} + \alpha \left(\overline{\psi}^{n-2} - 2\psi^{n-1} + \psi^n \right)$$

Second-order diffusion in time

Conservation of Mass: Mass Fixers

Some dynamical cores are not mass-conserving by design (but this property is needed for long term climate simulations)

One option: An **a posteriori mass fixer**

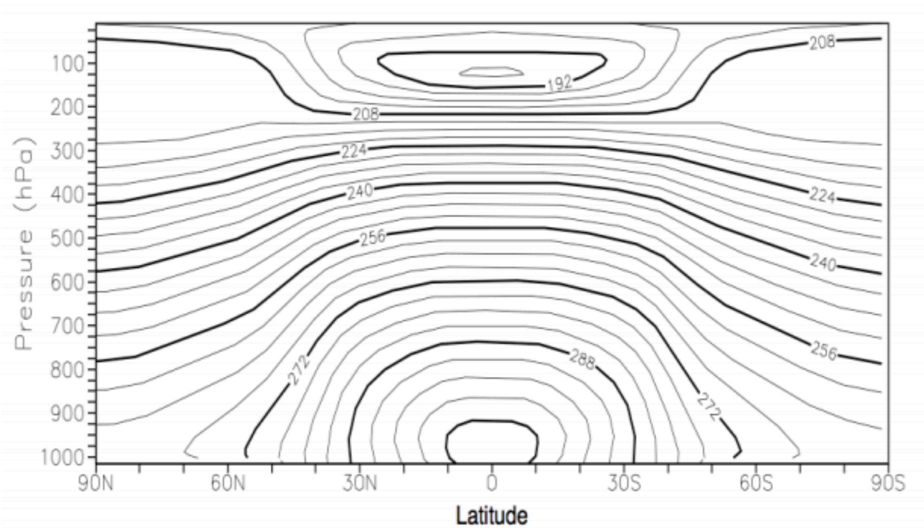
Basic idea: Adjust the mean value of the surface pressure (p_s) after each time step, so that total mass is conserved.

This technique **does not** alter the pressure gradients which are the driving force in the momentum equations.

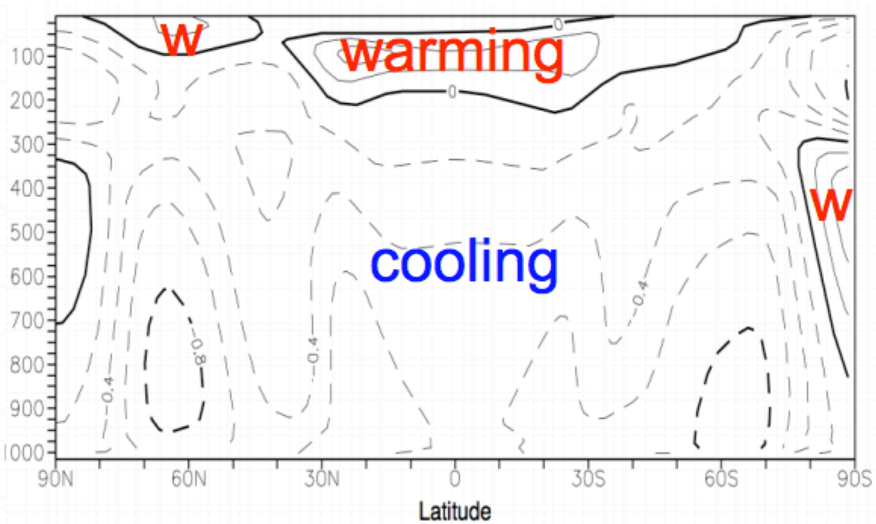
Conservation of Mass: Mass Fixers

- Weather forecast model IFS run with Held-Suarez test
- Compare the time-mean zonal-mean temperature of a run with and without mass fixer

Temperature
(without mass fixer)



Temperature
(with mass fixer - without)



Conservation of Total Energy

Total energy, which is not relevant for weather prediction models, is a relevant quantity to conserve in climate models.

When running for long times the violation of total energy conservation leads to artificial drifts in the climate system (e.g. ocean heat fluxes)

In nature:

- Conservation of total energy
- Energy lost by molecular diffusion (into eddies) provides heat

In atmospheric models:

- Energy is lost due to explicit or implicit diffusion
- Molecular diffusion is not represented on the model grid (spatial scales too big)
- An unstable numerical method might also lead to an increase in total energy

Therefore: Some models provide an a posteriori energy fixer that restores the conservation of total energy by modifying the temperature.

Conservation of Total Energy

Goal: Total energy at each time step should be constant

Compute residuals $RES = \hat{E}^{+} - E^{-}$

Compute total energy before (-) and after (+) each time step (here shown in hybrid pressure coordinates):

$$E^{+} = \int_A \left\{ \left[\sum_k \left(\frac{(\mathbf{v}_k^{+})^2}{2} + c_p \hat{T}_k^{+} \right) \left(p_0 \Delta A_k + p_s^{+} \Delta B_k \right) \right] + \Phi_s p_s^{+} \right\} dA$$

$$E^{-} = \int_A \left\{ \left[\sum_k \left(\frac{(\mathbf{v}_k^{-})^2}{2} + c_p \hat{T}_k^{-} \right) \left(p_0 \Delta A_k + p_s^{-} \Delta B_k \right) \right] + \Phi_s p_s^{-} \right\} dA$$

Idea: Correct the temperature field to achieve the conservation of total energy

How to return energy? Correct proportional to T? Correction is constant everywhere?

Summary

Diffusion and filters help maintain the numerical stability (removing aphysical effects such as numerical instability and replacing them with aphysical diffusion)

Some **diffusion** (either explicit or implicit) is **always needed** to prevent accumulation of energy at the smallest scale (due to truncated energy cascade)

But: Use these techniques selectively and know their consequences!

Word of caution: It is very easy to compute nice-looking smooth, highly diffusion, but very inaccurate solutions to the equations of motion.