ATM 265, Spring 2015
Lecture 5
Numerical Methods: Vertical Discretizations
April 15, 2019

Paul A. Ullrich (HH 251)
paullrich@ucdavis.edu

Outline

1. ***Unique Aspects of the Vertical***
2. ***Overview of Vertical Coordinate Systems***
3. ***Terrain-following Variations***
4. ***Vertical Computational Modes***
5. ***Semi-Lagrangian Layers***

Slides are based on Michael Toy's talk on vertical discretizations from the DCMIP workshop (2012)

Unique Aspects of the Vertical

Radius of the Earth
6371.22 km

Atmosphere Depth
100 km

Troposphere Depth
10 km

Mountain Height
5 km

Unique Aspects of the Vertical

Gravity acts in the vertical (let's look at the vertical momentum equation)

Typical scales (in m/s^2):

$$\frac{dw}{dt} - \frac{u^2 + v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} - g + 2\Omega u \cos\phi + \nu \nabla^2 w$$

10^{-7}	10^{-5}	10	10	10^{-3}	10^{-15}

Dominant balance is between these two terms.
Hydrostatic balance

Unique Aspects of the Vertical

For an approximately isothermal layer of the atmosphere,

Hydrostatic balance: $$-\frac{1}{\rho}\frac{\partial p}{\partial r} - g \approx 0$$

Ideal gas law: $$\rho = \frac{p}{R_d T_0}$$

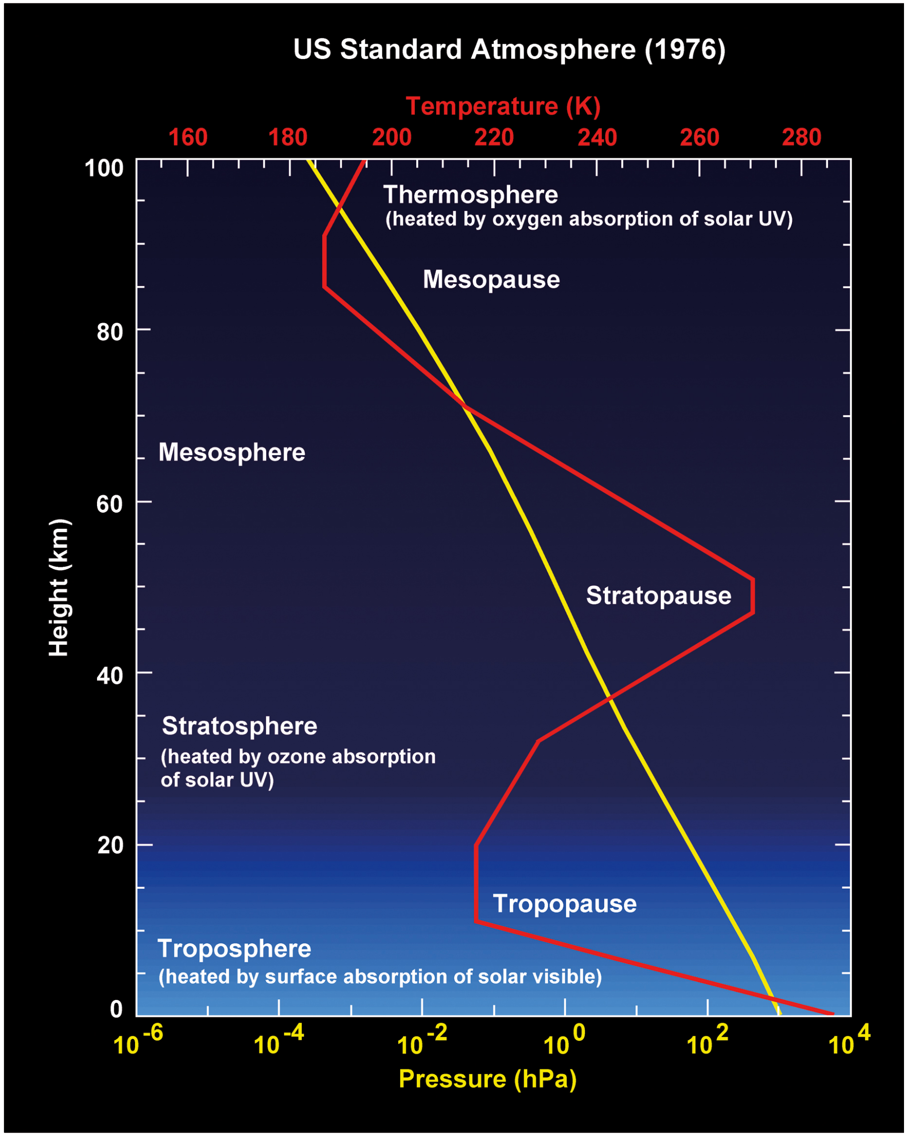
$$\frac{R_d T_0}{p}\frac{\partial p}{\partial r} = -g$$

Approx. Solution: $$p = p_0 \exp\left(\frac{-gz}{R_d T_0}\right) \qquad z \equiv r - a$$

Exponential decay of pressure and density with height! **Highly stratified!**

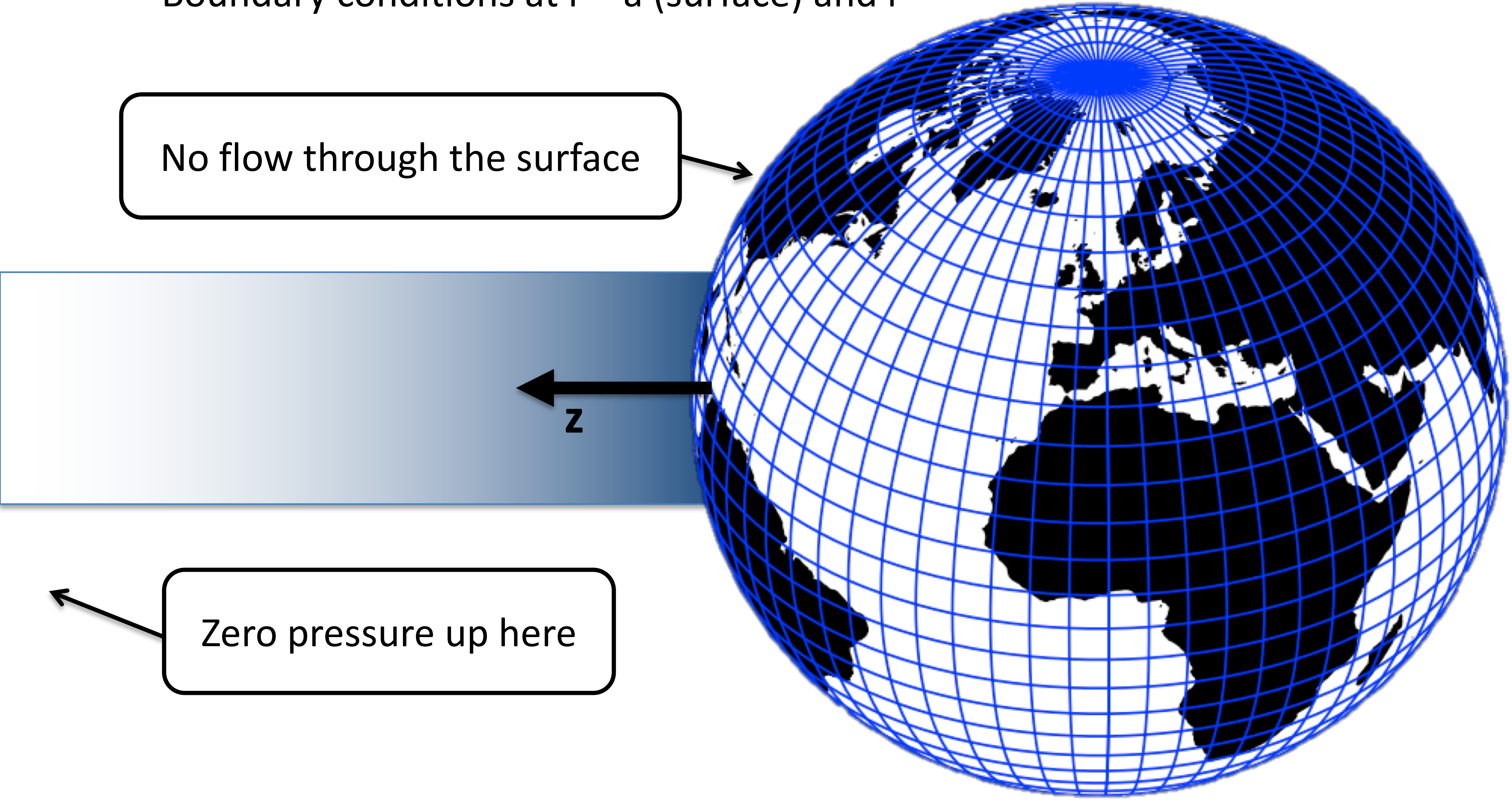
Unique Aspects of the Vertical

- **Gradients are much stronger in the vertical:** It's much colder 10km straight up (-50 C) and harder to breathe than 10km down the road.



Unique Aspects of the Vertical

- Boundary conditions at r = a (surface) and r = ∞



Unique Aspects of the Vertical

Many distinct physical processes:

- Convection
- Boundary layer
  - Viscous processes
- Radiation
- Waves
  - Troposphere-Stratosphere Interaction

Some not-so physical processes:

- Model-top sponge layer (numerical viscosity)

Numerical Considerations

- Choice of coordinate system
  - Height coordinates (Richardson 1922)
  - Pressure coordinates (Eliassen 1949)
  - Isentropic coordinates (Eliassen and Raustein 1968)
  - Mass coordinates (Laprise 1992)
  - Terrain-following or cut-cell?
- Ensure the hydrostatic relation is satisfied for a stratified atmosphere
- Staggering of variables?
- How to handle boundary conditions?

Non-Hydrostatic Primitive Equations

Five prognostic equations, one constraint equation

Shallow Atmosphere approximation assumed

$$\frac{du}{dt} - \frac{uv\tan\phi}{a} = -\frac{1}{\rho a\cos\phi}\left(\frac{\partial p}{\partial \lambda}\right)_z + 2\Omega v\sin\phi$$

$$\frac{dv}{dt} + \frac{u^2\tan\phi}{a} = -\frac{1}{\rho a}\left(\frac{\partial p}{\partial \phi}\right)_z - 2\Omega u\sin\phi$$

$$\frac{dw}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

$$\frac{d\rho}{dt} = -\rho\nabla\cdot\mathbf{u}$$

$$c_v\frac{dT}{dt} + p\frac{d\alpha}{dt} = J$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{u}_h\cdot\nabla_z + w\frac{\partial}{\partial z}$$

$$p = \rho R_d T$$

Hydrostatic Primitive Equations

Four prognostic equations, two constraint equation

$$\frac{du}{dt} - \frac{uv\tan\phi}{a} = -\frac{1}{\rho a\cos\phi}\left(\frac{\partial p}{\partial \lambda}\right)_z + 2\Omega v\sin\phi$$

$$\frac{dv}{dt} + \frac{u^2\tan\phi}{a} = -\frac{1}{\rho a}\left(\frac{\partial p}{\partial \phi}\right)_z - 2\Omega u\sin\phi$$

$$\frac{d\rho}{dt} = -\rho\nabla\cdot\mathbf{u}$$

$$c_v\frac{dT}{dt} + p\frac{d\alpha}{dt} = J$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{u}_h\cdot\nabla_z + w\frac{\partial}{\partial z}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

$$p = \rho R_d T$$

Z-Coordinates (Shallow Atmosphere)

Computing the vertical velocity $w \equiv \frac{dz}{dt}$ $\mathbf{u}_h \equiv (u, v)$

Non-hydrostatic (Predicted / Prognosed)

$$\frac{dw}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

Hydrostatic dynamical cores rarely use z-coordinates due to this mess

Hydrostatic (Diagnosed)

$$w = -\int_0^z \nabla \cdot \mathbf{u}_h dz - \frac{1}{\gamma}\int_0^z \frac{1}{p}(B + Q) dz + \frac{1}{c_p}\int_0^z \frac{J}{T} dz$$

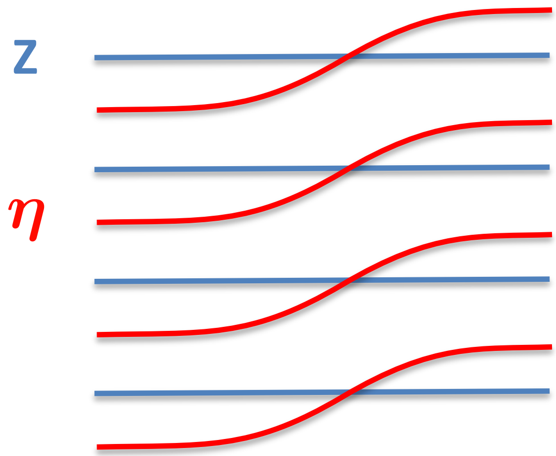
$$B = \frac{\frac{1}{\gamma}\int_0^{z_T} \frac{Q}{p} dz - \frac{1}{c_p}\int_0^{z_T} \frac{J}{T} dz + \int_0^{z_T} \nabla \cdot \mathbf{u}_h dz}{-\frac{1}{\gamma}\int_0^{z_T} \frac{dz}{p}}$$

$$Q = \mathbf{u}_h \cdot \nabla p - g \int_z^{z_T} \nabla \cdot (\rho \mathbf{u}_h) dz$$

Examples:
Kasahara and Washington (1967)
DeMaria (1995)

Vertical Coordinate Transforms

$$z(x,y,\eta,t) \longleftrightarrow \eta(x,y,z,t)$$



Transformation rules:

$$\frac{\partial}{\partial z} = \frac{\partial \eta}{\partial z}\frac{\partial}{\partial \eta}$$

$$\left(\frac{\partial}{\partial x}\right)_z = \left(\frac{\partial}{\partial x}\right)_\eta - \left(\frac{\partial z}{\partial x}\right)_\eta \frac{\partial \eta}{\partial z}\frac{\partial}{\partial \eta}$$

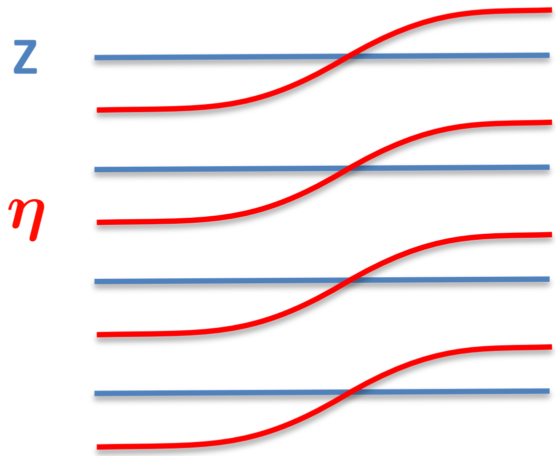
$$\left(\frac{\partial}{\partial y}\right)_z = \left(\frac{\partial}{\partial y}\right)_\eta - \left(\frac{\partial z}{\partial y}\right)_\eta \frac{\partial \eta}{\partial z}\frac{\partial}{\partial \eta}$$

New generalized vertical velocity:

$$\dot{\eta} \equiv \frac{d\eta}{dt} = \left(\frac{\partial \eta}{\partial t}\right)_z + \mathbf{u}_h \cdot \nabla_z \eta + w\frac{\partial \eta}{\partial z}$$

Vertical Coordinate Transforms

$$z(x, y, \eta, t) \longleftrightarrow \eta(x, y, z, t)$$



Pretty much any quantity can be chosen for your vertical coordinate, but it must be ***monotone***:

It must be either strictly increasing or decreasing as a function of height.

Eliminates options such as temperature from being used as a vertical coordinate.

Non-Hydrostatic Equations (General Coordinates)

Six prognostic equations, one constraint equation

$$\frac{du}{dt} - \frac{uv\tan\phi}{a} = \frac{1}{a\cos\phi}\left[-\frac{1}{\rho}\left(\frac{\partial p}{\partial \lambda}\right)_\eta + \frac{1}{m}\frac{\partial p}{\partial \eta}\left(\frac{\partial z}{\partial \lambda}\right)_\eta\right] + 2\Omega v \sin\phi$$

$$\frac{dv}{dt} + \frac{u^2\tan\phi}{a} = \frac{1}{a}\left[-\frac{1}{\rho}\left(\frac{\partial p}{\partial \phi}\right)_\eta + \frac{1}{m}\frac{\partial p}{\partial \eta}\left(\frac{\partial z}{\partial \phi}\right)_\eta\right] - 2\Omega u \sin\phi$$

$$\frac{dw}{dt} = -\frac{1}{m}\frac{\partial p}{\partial \eta} - g$$

$$\left(\frac{\partial m}{\partial t}\right)_\eta + \nabla_\eta \cdot (m\mathbf{u}_h) + \frac{\partial}{\partial \eta}(m\dot{\eta}) = 0$$

$$c_v\frac{dT}{dt} + p\frac{d\alpha}{dt} = J$$

$$\frac{dz}{dt} = w$$

$$m \equiv \rho\frac{\partial z}{\partial \eta}$$

New Mass Variable (pseudo-density)

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{u}_h \cdot \nabla_z + w\frac{\partial}{\partial z}$$

$$p = \rho R_d T$$

Non-Hydrostatic Equations

Six prognostic equations, one constraint equation

Two-term horizontal pressure gradient!

$$\frac{du}{dt} - \frac{uv\tan\phi}{a} = \frac{1}{a\cos\phi}\left[-\frac{1}{\rho}\left(\frac{\partial p}{\partial \lambda}\right)_\eta + \frac{1}{m}\frac{\partial p}{\partial \eta}\left(\frac{\partial z}{\partial \lambda}\right)_\eta\right] + 2\Omega v \sin\phi$$

$$\frac{dv}{dt} + \frac{u^2\tan\phi}{a} = \frac{1}{a}\left[-\frac{1}{\rho}\left(\frac{\partial p}{\partial \phi}\right)_\eta + \frac{1}{m}\frac{\partial p}{\partial \eta}\left(\frac{\partial z}{\partial \phi}\right)_\eta\right] - 2\Omega u \sin\phi$$

$$\frac{dw}{dt} = -\frac{1}{m}\frac{\partial p}{\partial \eta} - g$$

$$\left(\frac{\partial m}{\partial t}\right)_\eta + \nabla_\eta \cdot (m\mathbf{u}_h) + \frac{\partial}{\partial \eta}(m\dot{\eta}) = 0$$

$$c_v\frac{dT}{dt} + p\frac{d\alpha}{dt} = J$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{u}_h \cdot \nabla_\eta + \dot{\eta}\frac{\partial}{\partial \eta}$$

$$m \equiv \rho\frac{\partial z}{\partial \eta}$$

$$\frac{dz}{dt} = w$$

z is predicted!

$$p = \rho R_d T$$

Pressure Coordinates

$$\eta = p \qquad \omega \equiv \dot{\eta}$$

Non-Hydrostatic Equations

Vertical Momentum

$$\frac{dw}{dt} = -\frac{1}{m} - g$$

Horizontal pressure gradient

$$HPGF = \frac{1}{m}\nabla_p z$$

Continuity Equation

$$\left(\frac{\partial m}{\partial t}\right)_p + \nabla_p \cdot (m\mathbf{u}_h) + \frac{\partial}{\partial p}(m\omega) = 0$$

Vertical velocity difficult to diagnose…

Hydrostatic Equations

Layer mass is constant

$$m = -\frac{1}{g}$$

Single term pressure gradient

$$HPGF = -g\nabla_p z$$

Diagnostic continuity equation

$$\nabla_p \cdot (\mathbf{u}_h) + \frac{\partial \omega}{\partial p} = 0$$

$$\Rightarrow \quad \omega \equiv \dot{p} = -\int_0^p \nabla_p \cdot \mathbf{u}_h dp$$

Mass Coordinates

AKA Hydrostatic pressure coordinates (Laprise 1992)

$$\eta = \pi \qquad \pi(x,y,z,t) \equiv \int_z^{\infty} \rho(x,y,z',t) g dz'$$

Arises from the hydrostatic pressure equation: $\frac{\partial p}{\partial z} = -\rho g$

Represents the mass of air above a given height.

For a hydrostatically balanced atmosphere this is the pressure p.

Mass Coordinates

Non-Hydrostatic Equations

Pseudo-density

$$m = \rho \frac{dz}{d\pi} = -\frac{1}{g}$$

Horizontal pressure gradient

$$HPGF = -\frac{1}{\rho}\nabla_\pi p - \frac{\partial p}{\partial \pi}\nabla_\pi (gz)$$

Continuity Equation

$$\nabla_\pi \cdot \mathbf{u}_h + \frac{\partial \dot{\pi}}{\partial \pi} = 0$$

Hydrostatic Equations

Layer mass is constant

Same as Non-Hydrostatic

Double term pressure gradient

Same as Non-Hydrostatic $\left(\frac{\partial p}{\partial \pi} = 1\right)$

Diagnostic continuity equation

Same as Non-Hydrostatic

$$\Rightarrow \quad \dot{\pi} = -\int_0^\pi \nabla_\pi \cdot \mathbf{u}_h d\pi$$

Isentropic Coordinates

AKA Adiabatic coordinates

$$\eta = \theta$$

(Potential temperature)

$$\theta = T\left(\frac{p_0}{p}\right)^{R/c_p}$$

The vertical velocity is proportional to the diabatic heating:

$$\dot{\theta} \equiv \frac{d\theta}{dt} = \frac{J}{c_p}\left(\frac{p_0}{p}\right)^{R/c_p}$$

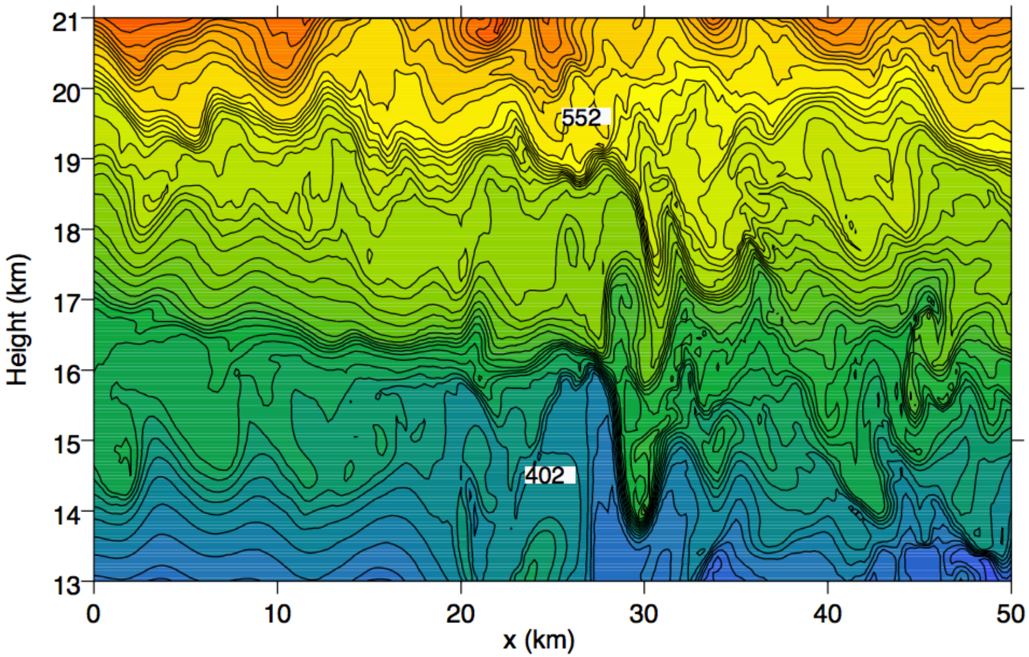
For an adiabatic atmosphere the “vertical motion” is zero and coordinate surfaces are material surfaces (a quasi-Lagrangain vertical coordinate)

This minimizes errors associated with vertical advection

Isentropic Coordinates

In non-hydrostatic models at high horizontal resolution, static instabilities and turbulence present a challenge since the coordinate loses the monotonicity property.

This is typically solved by ***hybridizing*** the coordinate with something which maintains monotonicity.



Vertical cross-section of isentropes associated with a breaking mountain wave.

Vertical Coordinates

Summary of four common vertical coordinates.

Coordinate	Non-Hydrostatic Models	Hydrostatic Models
Height (z)	Suitable	Not preferred (difficult to diagnose w)
Pressure (p)	Not preferred (difficult to diagnose ω)	Suitable
Mass (π)	Suitable	Suitable (identical to p coordinate)
Potential temperature (θ)	Suitable (some challenges)	Suitable

Representation of Topography

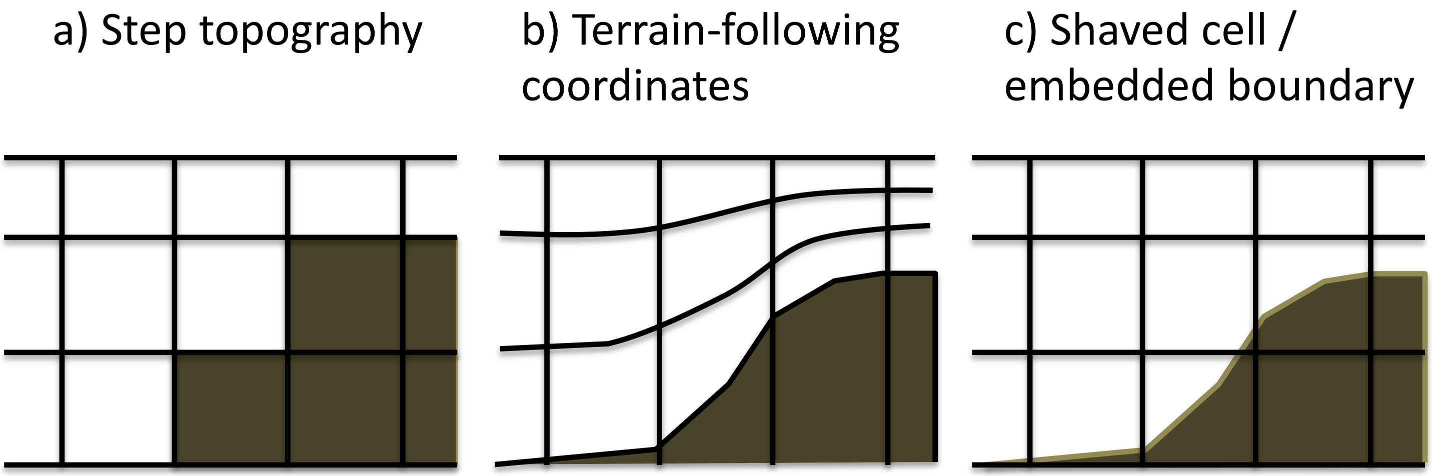


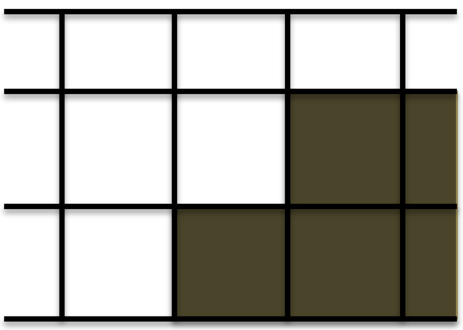
Figure: Adcroft et al. (1997)

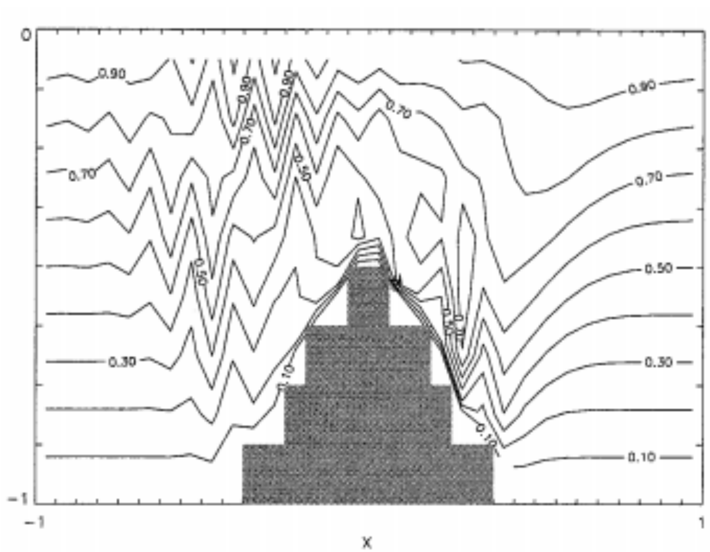
- Step topography is the easiest to implement, but is inaccurate and tends to produce significant spurious oscillations.

- Terrain-following coordinates have been the standard for atmospheric modeling systems, but issues with accurate computations of the horizontal pressure gradient force have led to interest in alternatives…

Representation of Topography

a) Step topography





Advantages:

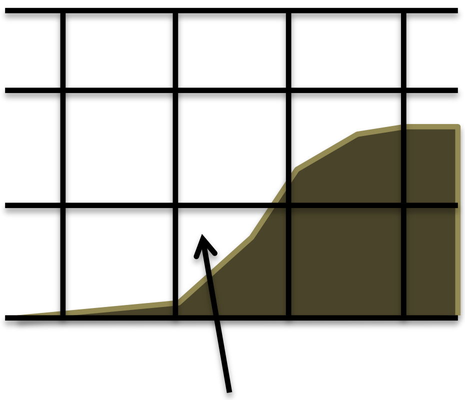
- No issues with small cells
- Accurate representation of horizontal pressure gradient force

Disadvantages:

- Poorly represents the underlying topography
- "Hard corners" create a lot of spurious noise

Representation of Topography

c) Shaved cell /
embedded boundary



Advantages:

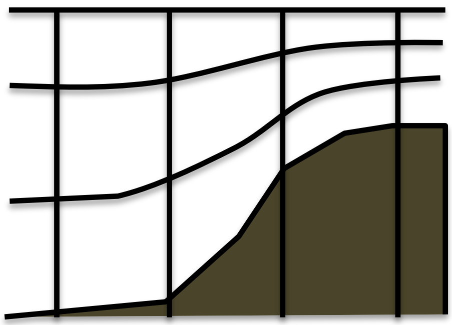
- Accurate treatment of pressure gradient force
- Accurate treatment of topography

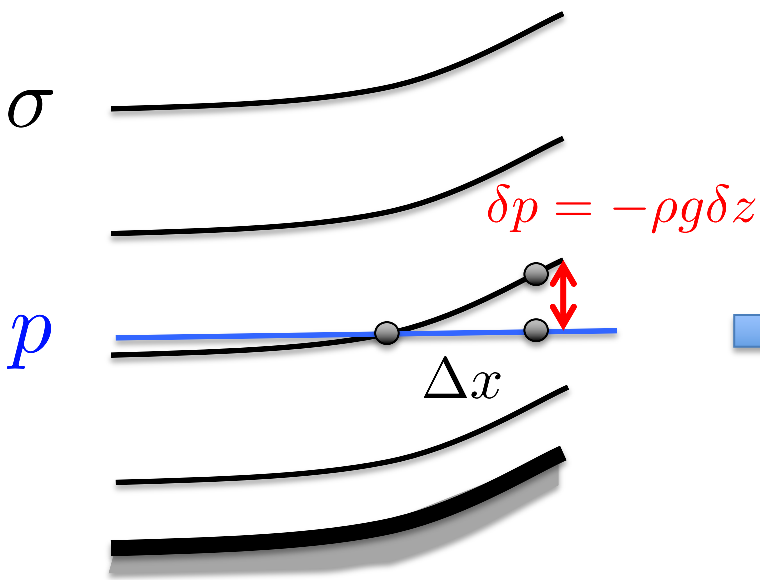
Disadvantages:

- Small grid cells can affect the maximum timestep size (CFL condition)

Representation of Topography

b) Terrain-following coordinates





Advantages:

- Topography absorbed into evolution equations
- Very accurate for smooth topography

Disadvantages:

- Poor representation of horizontal pressure-gradient force near steep topography

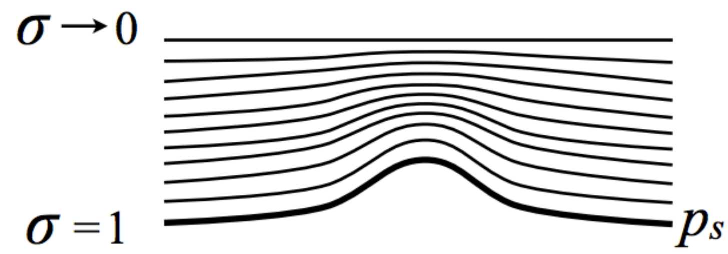
$$-g\nabla_p z = 0 = -g\nabla_\sigma z - \frac{1}{\rho}\nabla_\sigma p$$

The discrete form of these terms don't necessarily cancel!

Representation of Topography

$$\sigma \equiv \frac{p}{p_s}$$

Phillips (1957) Sigma coordinate



$$\eta_z \equiv \frac{z - z_s}{z_T - z_s}$$

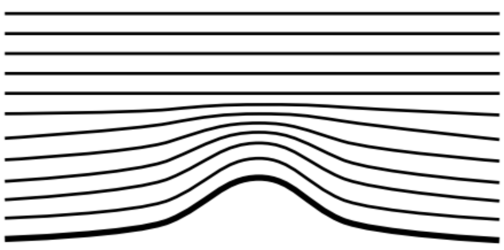
Gal-Chen and Somerville (1975)

$$\eta = \frac{\pi - \pi_T}{\pi_S - \pi_T}$$

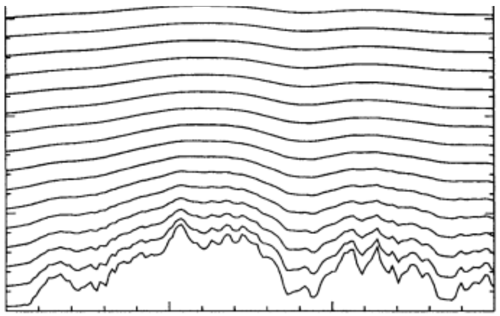
Skamarock and Klemp (2008)
Normalized hydrostatic pressure

$$\eta = \frac{p}{p_s} + \left(\frac{p}{p_s} - 1\right)\left(\frac{p}{p_s} - \frac{p}{p_0}\right)$$

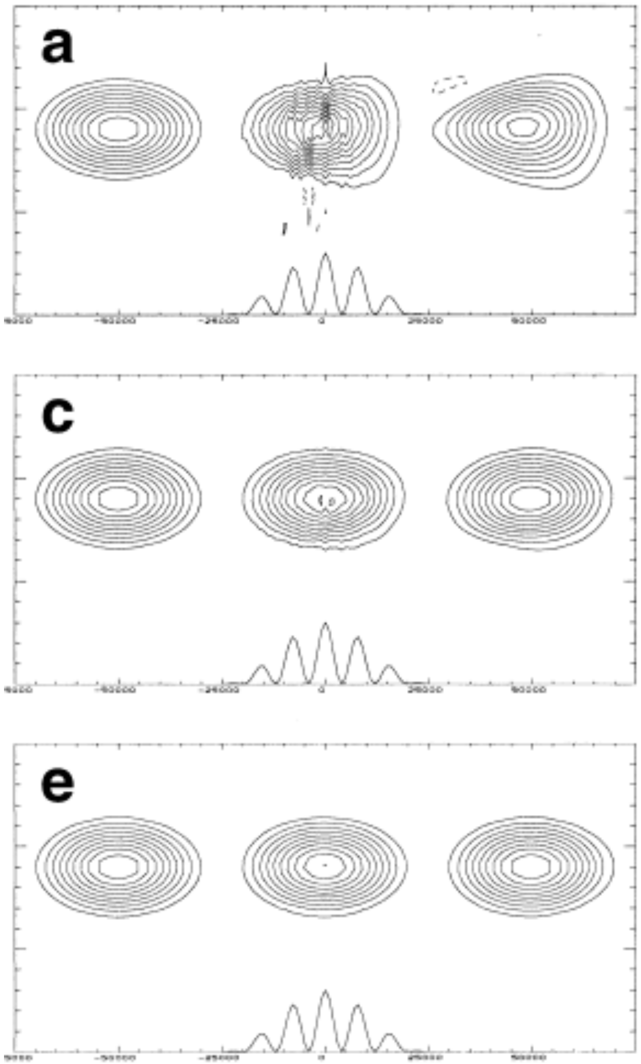
Simmons and Burridge (1981)
Hybrid coordinate



Schar et al. (2002) Hybrid coordinate



Representation of Topography



Second-Order Advection over Topography

Sigma Coordinate

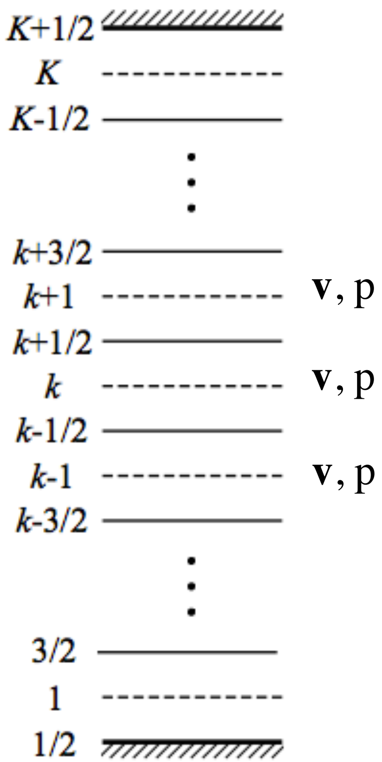
Simmons and Burridge (1981)
Hybrid coordinate

Schar (2002)
Hybrid coordinate

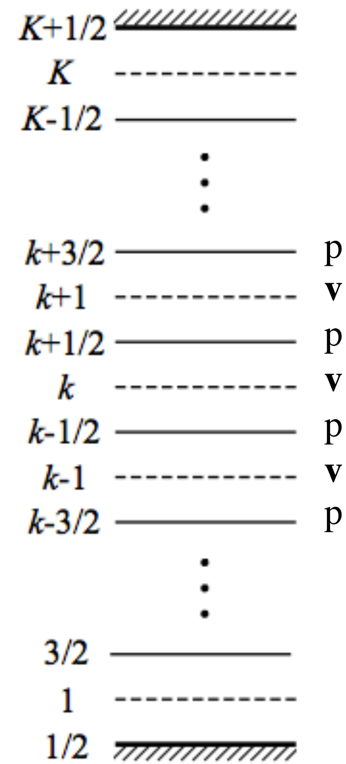
Source: Schar et al. (2002)

Vertical Staggering

Lorenz ("L") grid



Charney-Phillips ("CP") grid



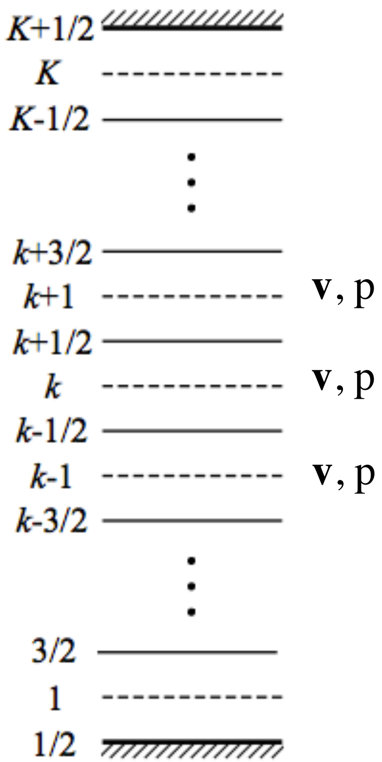
Location of levels and interfaces is identical. Only difference is staggering of velocity and potential temperature.

Like an Arakawa A-grid

Like an Arakawa C-grid

Vertical Staggering

Lorenz ("L") grid



Like an Arakawa A-grid

Linearized vertical velocity equation:

$$\frac{\partial w}{\partial t} + \frac{g}{\overline{\rho}}\rho + \frac{1}{\overline{\rho}}\frac{\partial p}{\partial z} = 0$$

Density perturbation (affects magnitude)

Ignore for now

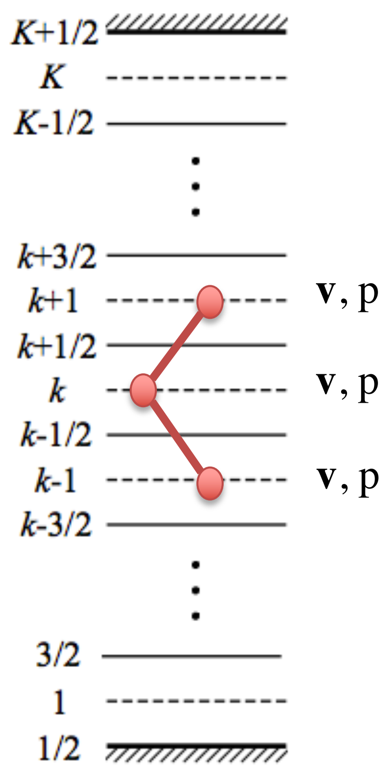
Pressure perturbation (affects propagation)

Discretization on Lorenz grid:

$$\frac{\partial w}{\partial t} + \frac{1}{\overline{\rho}}\left(\frac{p_{k+1} - p_{k-1}}{z_{k+1} - z_{k-1}}\right) = 0$$

Vertical Staggering: Lorenz

Lorenz ("L") grid



Like an Arakawa A-grid

Linearized vertical velocity equation:

$$\frac{\partial w}{\partial t} + \frac{g}{\overline{\rho}}\rho + \frac{1}{\overline{\rho}}\frac{\partial p}{\partial z} = 0$$

Discretization on Lorenz grid:

$$\frac{\partial w_k}{\partial t} + \frac{1}{\overline{\rho}}\left(\frac{p_{k+1} - p_{k-1}}{z_{k+1} - z_{k-1}}\right) = 0$$

Separation of odd/even modes:

- Vertical velocity evolution equation is unable to see $2\Delta z$ modes in the pressure field.
- Called a **"Computational Mode"**

Vertical Staggering: Charney-Phillips

Linearized vertical velocity equation:

$$\frac{\partial w}{\partial t} + \frac{g}{\bar{\rho}}\rho + \frac{1}{\bar{\rho}}\frac{\partial p}{\partial z} = 0$$

Discretization on Charney-Phillips grid:

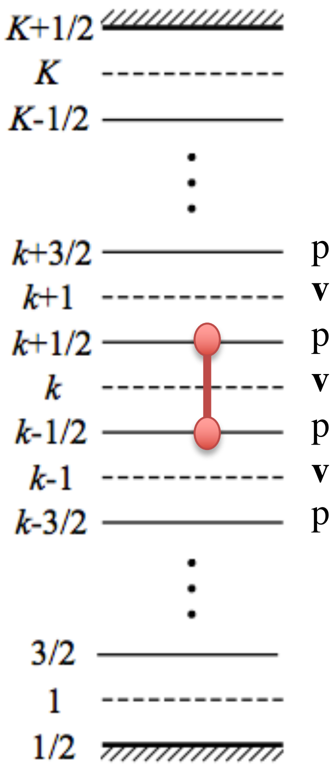
$$\frac{\partial w_k}{\partial t} + \frac{1}{\bar{\rho}}\left(\frac{p_{k+1/2} - p_{k-1/2}}{z_{k+1/2} - z_{k-1/2}}\right) = 0$$

No separation of odd/even modes.
No computational mode supported.

Also see:

- Tokioka (1978)
- Arakawa and Moorthi (1988)
- Arakawa and Konor (1996)

Charney-Phillips ("CP") grid



Like an Arakawa C-grid

Vertical Staggering

There are many possible choices of vertical staggering besides the ones chosen here:

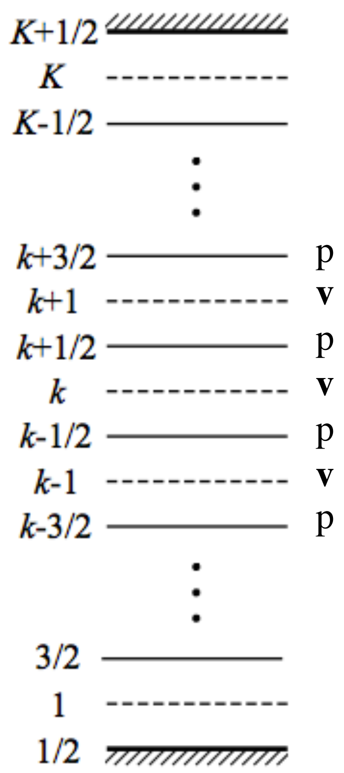
- For non-hydrostatic system there are 5 prognostic variables
- We can choose any two thermodynamic variables from ϱ, p, T, θ, etc.

Some have computational modes. Others do not.

- Accurate representation of waves (acoustic, inertia-gravity, Rossby) best achieved by minimizing finite differences over 2Δz.

- Analyzed by Thuburn and Woollings (2005) in three coordinate systems.

Charney-Phillips ("CP") grid



Semi-Lagrangian Coordinates

Instead of fixing the vertical coordinate, a common strategy is the use of semi-Lagrangian (or quasi-Lagrangian) vertical coordinates.

This is analogous to the semi-Lagrangian advection technique discussed last time, except only applied to the vertical. Remapping to a fixed Eulerian grid is typically performed every 15 or 30 minutes.

Introduced by Starr (1945)
In models:

- Skamarock (1998)
- He (2002)
- Lin (2004)
- Zängl (2007)
- Toy and Randall (2009)
- Toy (2011)

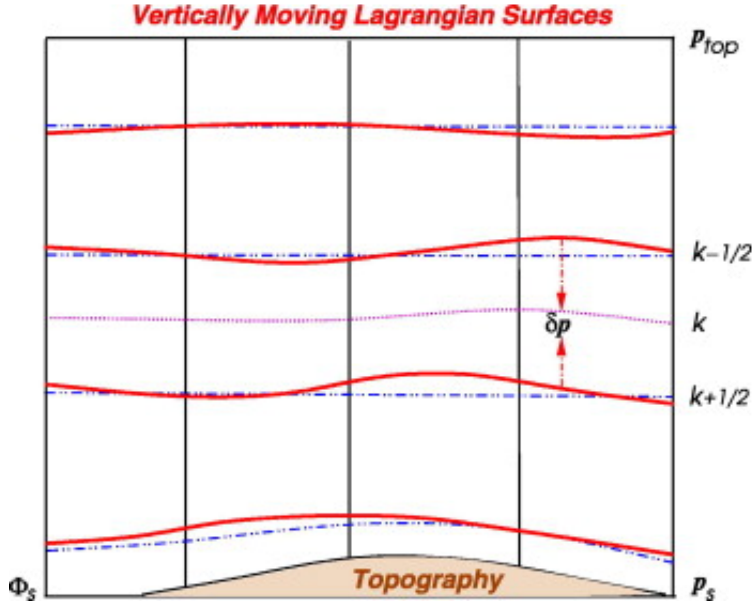


Figure: Nair et al. (2009)

Semi-Lagrangian Equations

Six prognostic equations, one constraint equation

$$\frac{du}{dt} - \frac{uv\tan\phi}{a} = \frac{1}{a\cos\phi}\left[-\frac{1}{\rho}\left(\frac{\partial p}{\partial \lambda}\right)_\eta + \frac{1}{m} \quad \cdots \quad \phi\right.$$

$$\frac{dv}{dt} + \frac{u^2\tan\phi}{a} = \frac{1}{a}\left[-\frac{1}{\rho}\left(\frac{\partial p}{\partial \phi}\right)_\eta + \frac{1}{m} \quad \cdots \quad \phi\right.$$

If layers are floating with the vertical velocity, then there is no vertical advection through layer surfaces.

$$\frac{dw}{dt} = -\frac{1}{m}\frac{\partial p}{\partial \eta} - g$$

$$\left(\frac{\partial m}{\partial t}\right)_\eta + \nabla_\eta \cdot (m\mathbf{u}_h) + \frac{\partial}{\partial \eta}(m\dot{\eta}) = 0$$

$$c_v\frac{dT}{dt} + p\frac{d\alpha}{dt} = J$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{u}_h \cdot \nabla_\eta + \dot{\eta}\frac{\partial}{\partial \eta}$$

$$\frac{dz}{dt} = w$$

z is predicted!

$$m \equiv \rho\frac{\partial z}{\partial \eta}$$

$$p = \rho R_d T$$

Semi-Lagrangian Coordinates

Layer thickness becomes a prognostic variable:

$$\frac{d(\delta p_k)}{dt} = \dot{\eta}_{k+1/2} - \dot{\eta}_{k-1/2}$$

Before layer thicknesses can become too thin / thick, they are remapped back to the fixed Eulerian grid.

The remapping procedure takes the place of vertical advection.

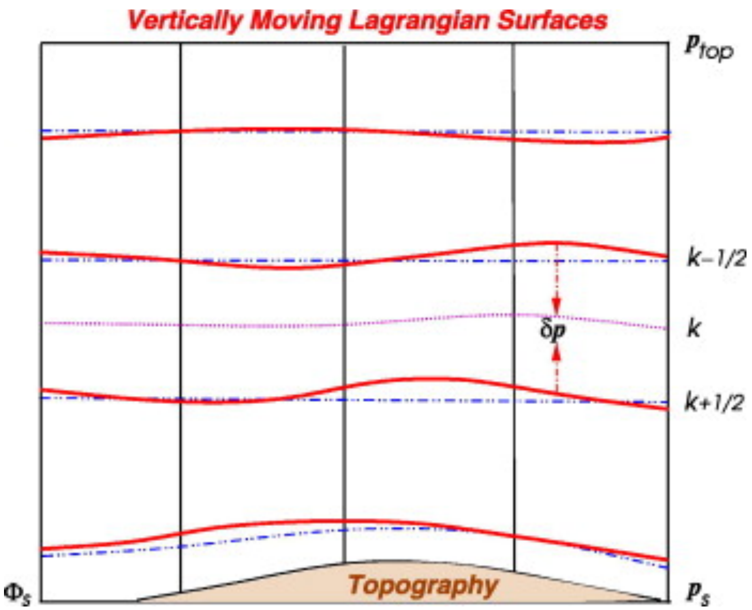


Figure: Nair et al. (2009)

Timestep Limitations (Non-Hydrostatic)

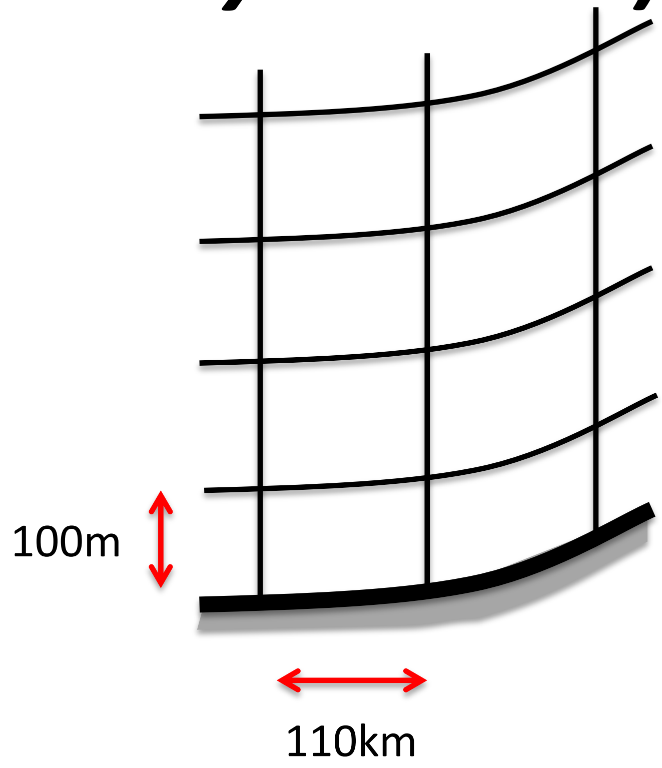
Recall that stability of an explicit numerical method requires a CFL condition to be satisfied. For example:

$$\left| \frac{u\Delta t}{\Delta x} \right| \leq 1$$

Here u is the maximum wave speed of the system, Δx is the grid spacing and Δt is the time step size.

In the atmosphere, the fastest waves are ***sound waves***, with a maximum speed of

$$u \approx \sqrt{\frac{\gamma p}{\rho}} \approx 342 \mathrm{m/s}$$



Only relevant for non-hydrostatic atmospheric models. Hydrostatic models do not support vertically propagating sound waves!

Timestep Limitations

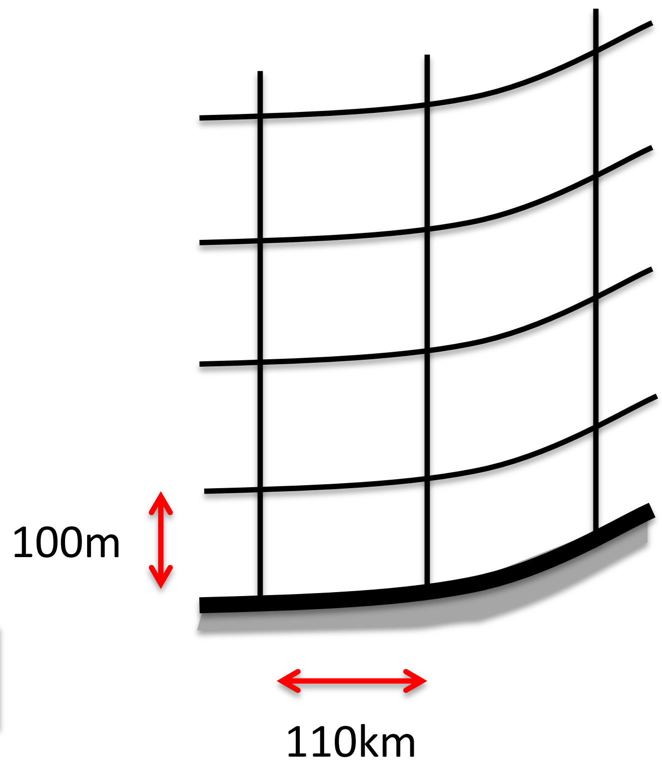
CFL Condition:

$$\left| \frac{u \Delta t}{\Delta x} \right| \leq 1$$

with $u \approx 342 \text{ m/s}$

for $\Delta x = 110 \text{ km}$ $\Delta t \leq$?

for $\Delta x = 100 \text{ m}$ $\Delta t \leq$?



In practice, limiting the time step size to be governed by the vertical coordinate is too severe! Need a better approach…

Overcoming Timestep Limitations

Filtered Equations:

- Anelastic approximation (Ogura and Phillips, 1962)
- Pseudo-Incompressible approximation (Durran, 1989)
- Unified approximation (Arakawa and Konor 2005)

(basic premise is to derive a new set of equations which do not contain sound waves and so maximum wave speed is limited by advection)

Numerical Methods:

- Implicit time stepping in the vertical