Atmospheric Waves
Chapter 6

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Part 5: Mixed Rossby-Gravity Waves

Trapped Waves

Trapped waves decay exponentially rapidly away from a boundary (such as the equator or a coastline). Trapped waves include:

- Equatorial (and coastal) Kelvin waves
- Equatorial Rossby (ER) waves
- Mixed Rossby-Gravity (MGR) waves at the equator
- Equatorial inertia-gravity waves

Equatorial Rossby and Mixed Rossby-Gravity Waves

- Shallow water equations, no topography
- Coriolis parameter at the equator is approximated by equatorial β-plane:

$$f \approx \beta y \approx \left(\frac{2\Omega}{a}\right) y$$

$$\frac{Du}{Dt} - \beta y v + \frac{\partial \Phi}{\partial x} = 0$$

$$\frac{Dv}{Dt} + \beta y u + \frac{\partial \Phi}{\partial y} = 0$$

$$\frac{D\Phi}{Dt} + \Phi\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

with $\Phi = gh$

Equatorial Rossby and Mixed Rossby-Gravity Waves

Linearize shallow water equations about a state at rest with mean height H

$$u = u', v = v', h = H + h'$$

$$\frac{\partial u'}{\partial t} - \beta y v' + \frac{\partial \Phi'}{\partial x} = 0 \qquad (1)$$

$$\frac{\partial v'}{\partial t} + \beta y u' + \frac{\partial \Phi'}{\partial y} = 0 \qquad (2)$$

$$\frac{\partial \Phi'}{\partial t} + gH \left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} \right) = 0 \qquad (3)$$

with $\Phi' = gh'$

Seek wave solutions of the form (allow amplitudes to vary in y):

$$(u', v', \Phi') = \left(\hat{u}(y), \hat{v}(y), \hat{\Phi}(y) \right) \exp(i(kx - \nu t))$$

Equatorial Rossby and Mixed Rossby-Gravity Waves

Yields the system

$$-i\nu\hat{u} - \beta y\hat{v} + ik\hat{\Phi} = 0 \qquad (1)$$

$$-i\nu\hat{v} + \beta y\hat{u} + \frac{\partial\hat{\Phi}}{\partial y} = 0 \qquad (2)$$

$$-i\nu\hat{\Phi} + gH\left(ik\hat{u} + \frac{\partial\hat{v}}{\partial y}\right) = 0 \qquad (3)$$

Solve (1) for $\hat{u}$:

$$\hat{u} = i\frac{\beta y}{\nu}\hat{v} + \frac{k}{\nu}\hat{\Phi} \qquad (4)$$

Equatorial Rossby and Mixed Rossby-Gravity Waves

Plug (4) into (2) and (3), rearrange terms

$$\left(\beta^2 y^2 - \nu^2\right) \hat{v} - ik\beta y \hat{\Phi} - i\nu \frac{\partial \hat{\Phi}}{\partial y} = 0 \qquad (5)$$

$$\left(\nu^2 - gHk^2\right) \hat{\Phi} + i\nu gH \left(\frac{\partial \hat{v}}{\partial y} - \frac{k}{\nu} \beta y \hat{v} \right) = 0 \qquad (6)$$

Solve (6) for $\hat{\Phi}$:

$$\hat{\Phi} = -\frac{i\nu gH}{(\nu^2 - gHk^2)} \left(\frac{\partial \hat{v}}{\partial y} - \frac{k}{\nu} \beta y \hat{v} \right) \qquad (7)$$

Equatorial Rossby and Mixed Rossby-Gravity Waves

Plug (7) into (5) and rearrange terms. Yields second-order differential equation for amplitude function

$$\frac{\partial^2 \hat{v}}{\partial y^2} + \left[\left(\frac{\nu^2}{gH} - k^2 - \frac{k}{\nu}\beta \right) - \frac{\beta^2 y^2}{gH} \right] \hat{v} = 0 \qquad (8)$$

More complicated equation than before since the **coefficient** in square brackets is **not constant** (depends on y)

Before discussing the solution in detail, let's first look at two asymptotic limits when either $H \rightarrow \infty$ or $\beta=0$

Equatorial Rossby and Mixed Rossby-Gravity Waves

Asymptotic limit H→∞

Means that the motion is non-divergent, consequence of the continuity equation (3):

$$\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) = -\frac{1}{gH}\frac{\partial \Phi'}{\partial t} \to 0 \quad \text{for} \quad H \to \infty$$

Equation (8) becomes

$$\frac{\partial^2 \hat{v}}{\partial y^2} + \left[-k^2 - \frac{k}{\nu}\beta\right]\hat{v} = 0 \qquad (9)$$

Equatorial Rossby and Mixed Rossby-Gravity Waves

Asymptotic limit H→∞

Asymptotic limit H→

$$\frac{\partial^2 \hat{v}}{\partial y^2} + \left[-k^2 - \frac{k}{\nu}\beta\right]\hat{v} = 0 \qquad (9)$$

Solutions for the amplitude function $\hat{v}$ in (9) exists of the form:

$$\hat{v} = v_0 \exp(i\ell y)$$

Provided that the frequency ν satisfies the Rossby wave dispersion relationship:

$$\boxed{\nu = \frac{-\beta k}{k^2 + \ell^2}}$$

It shows that for non-divergent barotropic flow, equatorial dynamics are in no way special Earth's rotation enters in form of β, not f: **Rossby wave**

Equatorial Rossby and Mixed Rossby-Gravity Waves

Asymptotic limit $\beta=0$

All influence of rotation is eliminated. Due to linearity, solutions of (9) exists of the form $\hat{v} = v_0 \exp(i\ell y)$

Equation (8) reduces to the shallow water gravity model

$$\frac{\partial^2 \hat{v}}{\partial y^2} + \left[\frac{\nu^2}{gH} - k^2 \right] \hat{v} = 0 \qquad (10)$$

Non-trivial solutions if frequency ν satisfies

$$\nu = \pm\sqrt{gH(k^2 + \ell^2)}$$

Pure gravity wave response (eastward and westward)

Equatorial Rossby and Mixed Rossby-Gravity Waves

In the general case with

$$\frac{\partial^2 \hat{v}}{\partial y^2} + \left[\left(\frac{\nu^2}{gH} - k^2 - \frac{k}{\nu}\beta \right) - \frac{\beta^2 y^2}{gH} \right] \hat{v} = 0 \qquad (8)$$

we seek solutions for the meridional distribution of $\hat{v}$ with boundary condition

$$\hat{v} \to 0 \quad \text{for} \quad |y| \to \infty$$

For small y the coefficient in square brackets is positive and solutions oscillate in y. For large y the coefficient is negative, solutions either grow or decay.

Only the **decaying** solution satisfies boundary condition.

Equatorial Rossby and Mixed Rossby-Gravity Waves

Matsuno (1966) showed that decaying solutions only exist if the constant part of the coefficient in square brackets satisfies the relationship

$$\frac{\sqrt{gH}}{\beta}\left(\frac{\nu^2}{gH} - k^2 - \frac{k}{\nu}\beta\right) = 2n + 1 \qquad n = 0, 1, 2, \ldots$$

This is a **cubic** dispersion relation for frequency ν.

It determines the allowed frequencies ν of equatorially trapped waves for the zonal wavenumber k and meridional mode number n.

Equatorial Rossby and Mixed Rossby-Gravity Waves

In (8) replace y by the nondimensional meridional coordinate

$$\xi = \left[\left(\frac{\beta}{\sqrt{gH}} \right)^{1/2} y \right]$$

Then the solution has the form $\hat{v}(\xi) = v_0 H_n(\xi) \exp\left(-\frac{\xi^2}{2} \right)$

Where v_0 is a constant and $H_n(\xi)$ designates the n^{th} *Hermite polynomial.* The first few Hermite polynomials have the form

$$H_0(\xi) = 1, \qquad H_1(\xi) = 2\xi, \qquad H_2(\xi) = 4\xi^2 - 2$$

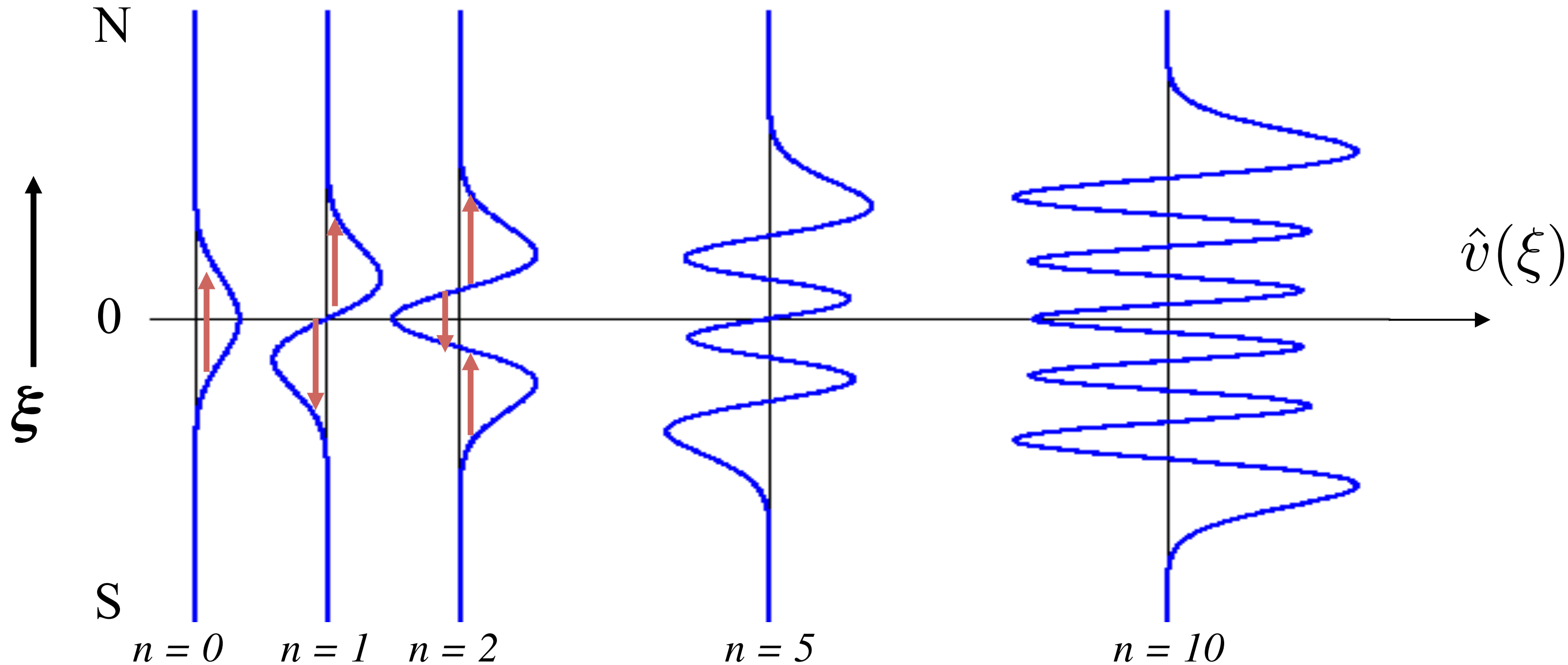
Index *n* corresponds to the number of N-S roots in v.

Hermite Polynomials

Meridional (N-S) structure of the amplitude function:

$$\hat{v}(\xi) = v_0 H_n(\xi) \exp\left(-\frac{\xi^2}{2}\right) \qquad n = 0, 1, 2, \ldots$$

Hermite polynomials: $H_n(\xi) = (-1)^n \exp(\xi^2) \left[\frac{d^n}{d\xi^n}\left(\exp\left(\xi^{-2}\right)\right)\right]$



Equatorial Rossby and Mixed Rossby-Gravity Waves

In general, the three ν solutions of

$$\frac{\sqrt{gH}}{\beta}\left(\frac{\nu^2}{gH} - k^2 - \frac{k}{\nu}\beta\right) = 2n+1 \qquad n = 0, 1, 2, \ldots$$

can be interpreted as an equatorially trapped

1) Eastward moving inertia-gravity wave
2) Westward moving inertia-gravity wave
3) Westward moving equatorial Rossby wave

However, the case $n = 0$ must be treated separately: For $n = 0$ the meridional velocity perturbation has a Gaussian distribution centered at the equator

Equatorial Rossby and Mixed Rossby-Gravity Waves

For *n* = 0 (special case) the dispersion relation factors as

$$\left(\frac{\nu}{\sqrt{gH}} - k - \frac{\beta}{\nu}\right)\left(\frac{\nu}{\sqrt{gH}} + k\right) = 0$$

The root $\nu = -k\sqrt{gH}$ (westward propagating gravity wave) is not permitted, violates an earlier assumptions when eliminating $\hat{\Phi}$

The two other roots are

$$\nu = k\sqrt{gH}\left[\frac{1}{2} \pm \sqrt{1 + \frac{4\beta}{k^2\sqrt{gH}}}\right]$$

Equatorial Rossby and Mixed Rossby-Gravity Waves

The **positive** root corresponds to an **eastward moving** equatorial **inertia-gravity** wave

The **negative** root corresponds to a **westward moving** wave, which resembles

- An equatorial (westward moving) **inertia-gravity (IG)** wave for long zonal scale ($k \rightarrow 0$)
- An **equatorial Rossby (ER)** wave for zonal scales characteristic of synoptic-scale disturbances

This mode ($n=0$, westward moving) is therefore called **Mixed Rossby-Gravity (MRG)** wave.

Equatorial Mixed Rossby-Gravity Waves

Plane view of horizontal velocity and height perturbations in the (*n=0*) Mixed Rossby-Gravity (MRG) wave, propagates westward