Atmospheric Waves
Chapter 6

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Part 3: Internal Gravity Waves

Internal Gravity Waves

(Internal) gravity waves are generated by flow over topography, thunderstorms, other mesoscale systems

(Internal) gravity wave effects are: cloud streets, lenticularis clouds, rapid local temperature and pressure changes, clear air turbulence

They carry energy and momentum into the middle atmosphere (stratosphere, mesosphere)

Internal Gravity Waves

Atmospheric gravity waves can only exist when the atmosphere is stably stratified ($N^2 > 0$)

Buoyancy force is the restoring force for gravity waves

In an incompressible fluid (e.g. ocean) gravity waves travel primarily in the horizontal plane, since reflection at lower and upper boundary occurs for vertically traveling waves.

In the atmosphere: Due to compressibility and stratification, gravity waves may propagate vertically and horizontally.

Topographic Gravity Waves

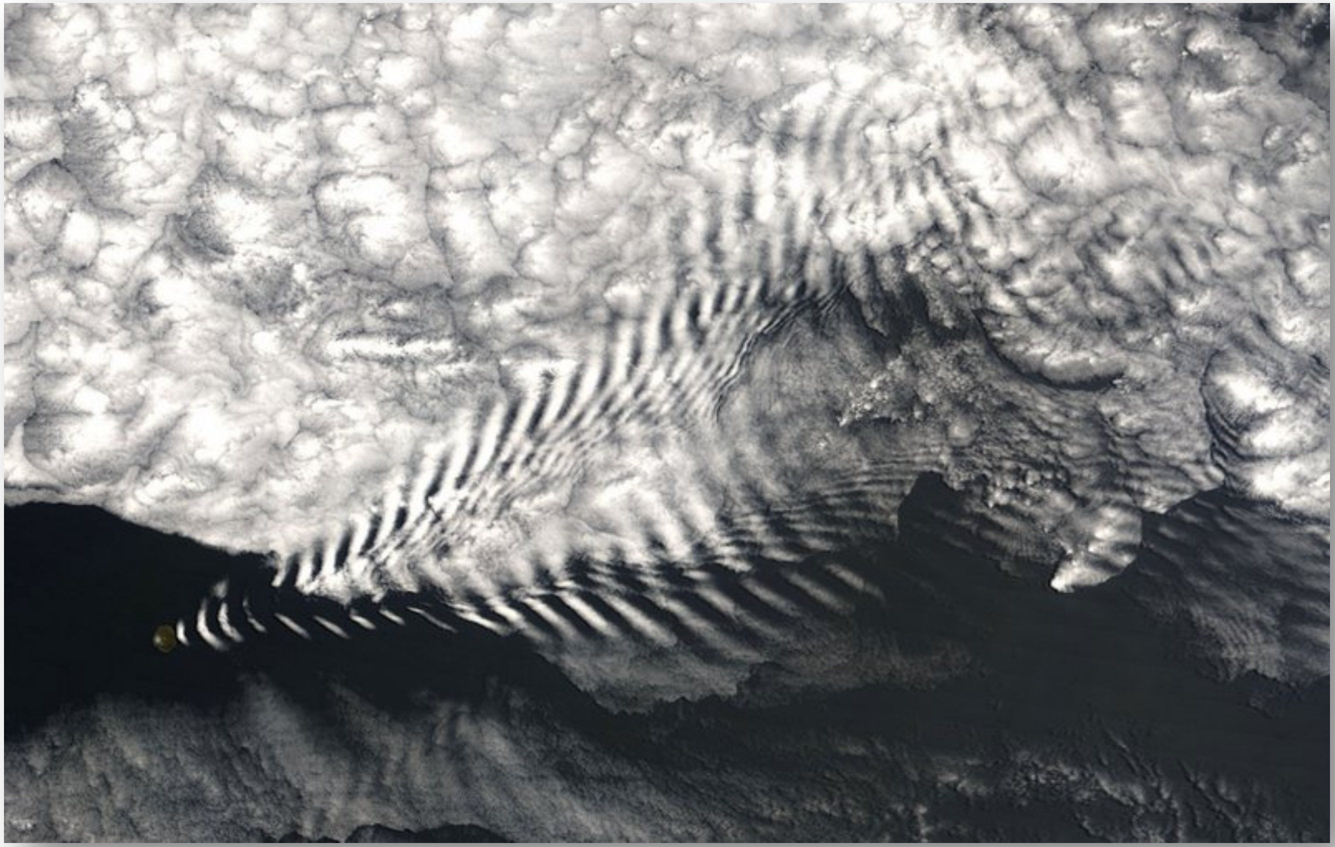


Figure: Air moving over a mountain on an island triggers waves on the downwind (lee) side of the mountain, indicated by the cloud pattern

Examples

Undular Bore

Examples

Mountain-generated waves (lee waves)

Internal Gravity Waves

Consider pure internal gravity waves:
No rotation, no friction

Can be studied with the basic equations for 2D (x-z plane) motions with Boussinesq approximation (almost incompressible)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial z} + g = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial x} + w\frac{\partial \theta}{\partial z} = 0$$

Four equations in five unknowns

Internal Gravity Waves

Equations must be closed via an equation of state, such as the potential temperature equation:

$$\theta = T\left(\frac{p_0}{p}\right)^{\frac{R_d}{c_p}} = \frac{p}{\rho R_d}\left(\frac{p_0}{p}\right)^{\frac{R_d}{c_p}} = R_d p^{\frac{c_v}{c_p}} \rho^{-1} p_0^{\frac{R_d}{c_p}}$$

Take logarithms of both sides:

$$\ln\theta = \frac{c_v}{c_p}\ln p - \ln\rho + \text{constant}$$

IGWs: Derivation (1)

Linearize the basic equations by letting

$$\rho = \rho_0 + \rho'$$
$$p = \overline{p}(z) + p'$$
$$\theta = \overline{\theta}(z) + \theta'$$
$$u = \overline{u} + u'$$
$$w = w'$$

The basic state must satisfy:

$$\frac{d\overline{p}}{dz} = -g\rho_0$$

$$\ln \overline{\theta} = \frac{c_v}{c_p} \ln \overline{p} - \ln \rho_0 + \text{constant}$$

Hydrostatic Balance

Definition of Potential Temperature

IGWs: Derivation (2)

Linearize the material derivative terms:

$$\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} &= \frac{\partial}{\partial t}(\underbrace{\cancel{\overline{u}}}^{\text{Constant}} + u') + (\overline{u} + u')\frac{\partial}{\partial x}(\underbrace{\cancel{\overline{u}}}^{\text{Constant}} + u') + w'\frac{\partial}{\partial z}(\underbrace{\cancel{\overline{u}}}^{\text{Constant}} + u'), \\
&= \frac{\partial u'}{\partial t} + \overline{u}\frac{\partial u'}{\partial x} + \underbrace{\cancel{u'\frac{\partial u'}{\partial x}}}_{\text{Small}} + \underbrace{\cancel{w'\frac{\partial u'}{\partial z}}}_{\text{Small}}, \\
&= \frac{\partial u'}{\partial t} + \overline{u}\frac{\partial u'}{\partial x}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} &= \frac{\partial w'}{\partial t} + (\overline{u} + \overbrace{\cancel{u'}}^{\text{Small}})\frac{\partial w'}{\partial x} + \overbrace{\cancel{w'\frac{\partial w'}{\partial z}}}^{\text{Small}}, \\
&= \frac{\partial w'}{\partial t} + \overline{u}\frac{\partial w'}{\partial x}
\end{aligned}$$

IGWs: Derivation (2)

Linearize the thermodynamic equation:

No time dependence

No x dependence

$$\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial x} + w\frac{\partial \theta}{\partial z} = \frac{\partial}{\partial t}(\cancel{\overline{\theta}} + \theta') + (\overline{u} + u')\frac{\partial}{\partial x}(\cancel{\overline{\theta}} + \theta') + w'\frac{\partial}{\partial z}(\overline{\theta} + \theta'),$$

$$= \frac{\partial \theta'}{\partial t} + \overline{u}\frac{\partial \theta'}{\partial x} + \cancel{u'\frac{\partial \theta'}{\partial x}} + w'\frac{\partial \overline{\theta}}{\partial z} + \cancel{w'\frac{\partial \theta'}{\partial z}},$$

Small

$$= \frac{\partial \theta'}{\partial t} + \overline{u}\frac{\partial \theta'}{\partial x} + w'\frac{\partial \overline{\theta}}{\partial z}$$

IGWs: Derivation (3)

Linearize $$\frac{1}{\rho}\frac{\partial p}{\partial z} + g = \frac{1}{\rho_0 + \rho'}\left(\frac{\partial \overline{p}}{\partial z} + \frac{\partial p'}{\partial z}\right) + g$$

Remember some calculus:

For $\dfrac{\rho'}{\rho_0} \ll 1$
$$\frac{1}{\rho_0 + \rho'} = \frac{1}{\rho_0}\frac{1}{\left(1 + \frac{\rho'}{\rho_0}\right)} \approx \frac{1}{\rho_0}\left(1 - \frac{\rho'}{\rho_0}\right)$$

Linearization yields

$$\frac{1}{\rho}\frac{\partial p}{\partial z} + g \approx \frac{1}{\rho_0}\frac{\partial \overline{p}}{\partial z}\left(1 - \frac{\rho'}{\rho_0}\right) + \frac{1}{\rho_0}\frac{\partial p'}{\partial z} + g = \frac{1}{\rho_0}\frac{\partial p'}{\partial z} + \frac{\rho'}{\rho_0} g$$

Use $\dfrac{1}{\rho_0}\dfrac{\partial \overline{p}}{\partial z} = -g$

IGWs: Derivation (3)

Potential temperature equation:

$$\ln \theta = \frac{c_v}{c_p} \ln p - \ln \rho + \text{constant}$$

Linearize via $\theta = \overline{\theta} + \theta', \quad p = \overline{p} + p', \quad \rho = \rho_0 + \rho'$

$$\Rightarrow \quad \ln \left[\overline{\theta} \left(1 + \frac{\theta'}{\overline{\theta}} \right) \right] = \frac{1}{\gamma} \ln \left[\overline{p} \left(1 + \frac{p'}{\overline{p}} \right) \right] - \ln \left[\rho_0 \left(1 + \frac{\rho'}{\rho_0} \right) \right] + \text{constant}$$

with $\gamma = \frac{c_p}{c_v}$

IGWs: Derivation (3)

Again, remember some calculus:

$$\ln(ab) = \ln a + \ln b$$
$$\ln(1+\epsilon) \approx \epsilon \qquad\qquad \epsilon \ll 1$$

Apply these rules to potential temperature equation:

$$\ln\left[\overline{\theta}\left(1+\frac{\theta'}{\overline{\theta}}\right)\right] = \frac{1}{\gamma}\ln\left[\overline{p}\left(1+\frac{p'}{\overline{p}}\right)\right] - \ln\left[\rho_0\left(1+\frac{\rho'}{\rho_0}\right)\right] + \mathrm{constant}$$

To obtain linearized potential temperature equation:

$$\frac{\theta'}{\overline{\theta}} \approx \frac{1}{\gamma}\frac{p'}{\overline{p}} - \frac{\rho'}{\rho_0} \quad\Rightarrow\quad \rho' \approx -\rho_0\frac{\theta'}{\overline{\theta}} + \frac{p'}{c_s^2}$$

Speed of sound

$$c_s^2 = \frac{\overline{p}\gamma}{\rho_0}$$

IGWs: Derivation (3)

In gravity waves, density fluctuations due to pressure changes are small compared to those due to temperature changes:

$$\left| \rho_0 \frac{\theta'}{\overline{\theta}} \right| \gg \left| \frac{p'}{c_s^2} \right|$$

Therefore, to a first approximation, the potential temperature equation becomes:

$$\boxed{\frac{\theta'}{\overline{\theta}} \approx -\frac{\rho'}{\rho_0}}$$

IGWs: Derivation (4)

The linearized equation set for internal gravity waves becomes

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right) u' + \frac{1}{\rho_0}\frac{\partial p'}{\partial x} = 0 \qquad (1)$$

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right) w' + \frac{1}{\rho_0}\frac{\partial p'}{\partial z} - \frac{\theta'}{\overline{\theta}} g = 0 \qquad (2)$$

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0 \qquad (3)$$

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right) \theta' + w'\frac{\partial \overline{\theta}}{\partial z} = 0 \qquad (4)$$

IGWs: Derivation (5)

Eliminate p' by computing $\dfrac{\partial(2)}{\partial x} - \dfrac{\partial(1)}{\partial z}$

Leads to

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)\left(\frac{\partial w'}{\partial x} - \frac{\partial u'}{\partial z}\right) - \frac{g}{\overline{\theta}}\frac{\partial \theta'}{\partial x} = 0 \qquad (5)$$

Now eliminate u' from (5) by computing

$$\frac{\partial(5)}{\partial x} + \left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)\frac{\partial(3)}{\partial z}$$

Leads to the equation:

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)\left(\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial z^2}\right) - \frac{g}{\overline{\theta}}\frac{\partial^2 \theta'}{\partial x^2} = 0 \qquad (6)$$

IGWs: Derivation (5)

Eliminate θ' from (6) by computing

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)(6) + \frac{g}{\overline{\theta}}\frac{\partial^2 (4)}{\partial x^2}$$

Yields single equation for w'

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)^2 \left(\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial z^2}\right) + \boxed{\frac{g}{\overline{\theta}}\frac{\partial \overline{\theta}}{\partial z}}\frac{\partial^2 w'}{\partial x^2} = 0$$

Square of Brunt-Väisälä frequency:
$\mathcal{N}^2$ assume to be constant

IGWs: Derivation (6)

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)^2 \left(\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial z^2}\right) + \mathcal{N}^2 \frac{\partial^2 w'}{\partial x^2} = 0$$

Consider **wave-like solutions**:

$$\hat{w} = w_r + i w_i \quad m = m_r + i m_i \quad \phi = kx + mz - \nu t$$

$$\begin{aligned} w' &= \mathrm{Re}\left[\hat{w}\exp(i\phi)\right] \\ &= \mathrm{Re}\left[(w_r + i w_i)\exp(i(kx + m_r z - \nu t))\exp(-m_i z)\right] \\ &= \left[w_r \cos(\mathrm{Re}(\phi)) - w_i \sin(\mathrm{Re}(\phi))\right]\exp(-m_i z) \end{aligned}$$

Horizontal wavenumber *k* is real, solution is always sinusoidal

IGWs: Wavenumbers

$$m = m_r + i m_i$$

$$w' = [w_r \cos(\mathrm{Re}(\phi)) - w_i \sin(\mathrm{Re}(\phi))] \exp(-m_i z)$$

The vertical wavenumber m is complex

- Real part m_r describes the sinusoidal variation in z
- Imaginary part m_i describes the exponential decay or growth of the wave amplitude in z, depending on the sign of m_i (positive / negative).

If m is real, total wave number may be regarded as a vector directed perpendicular to line of constant phase, and in the direction of phase increase.

$$\boldsymbol{\kappa} = k\mathbf{i} + m\mathbf{k} \quad \text{with} \quad k = \frac{2\pi}{L_x} \quad m = \frac{2\pi}{L_z}$$

IGWs: Dispersion

Insert $w' = \mathrm{Re}\left[\hat{w}\exp(i\phi)\right]$

into $$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)^2 \left(\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial z^2}\right) + \mathcal{N}^2 \frac{\partial^2 w'}{\partial x^2} = 0$$

IGW Dispersion Relation

$$(\nu - \overline{u}k)^2(k^2 + m^2) - \mathcal{N}^2 k^2 = 0$$

IGW Frequency

$$\nu = \overline{u}k \pm \frac{\mathcal{N}k}{\sqrt{k^2 + m^2}} = \overline{u}k \pm \frac{\mathcal{N}k}{\kappa}$$

Relative to the mean wind:

- Positive (+) sign means eastward phase propagation
- Negative (-) sign means westward phase propagation

IGWs: Frequency

If we let $k > 0$ and $m < 0$, then lines of constant phase $\phi = (kx + mz)$ tilt eastward with increasing height

The choice of the positive root

$$\nu = \overline{u}k \pm \frac{\mathcal{N}k}{\sqrt{k^2 + m^2}} = \overline{u}k \overset{+}{\pm} \frac{\mathcal{N}k}{\kappa}$$

then corresponds to an eastward and downward phase propagation (relative to the mean flow).

The zonal and vertical phase speeds (positive root) are

$$c_x = \frac{\nu}{k} = \overline{u} + \frac{\mathcal{N}}{\sqrt{k^2 + m^2}} \qquad c_z = \frac{\nu}{m} = \frac{\overline{u}k}{m} + \frac{\mathcal{N}k}{m\sqrt{k^2 + m^2}}$$

IGWs: Phase Relationships

Solutions for the other perturbation fields u', p', θ' can be obtained by substituting w' back into original linearized equations (let $k > 0$ and $m < 0$)

$$\frac{\partial u'}{\partial x} = -\frac{\partial w'}{\partial z} \quad \Longrightarrow \quad u' = -\frac{m}{k} w'$$

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right) u' = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x} \quad \Longrightarrow \quad p' = \rho_0 \frac{(\nu - k\overline{u})}{k} u'$$

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right) \theta' = -w'\frac{\partial \overline{\theta}}{\partial z} \quad \Longrightarrow \quad \theta' = \frac{i}{(k\overline{u} - \nu)}\frac{\partial \overline{\theta}}{\partial z} w'$$

Observe u', w', p' are exactly in phase, with $\nu - k\overline{u} > 0$

However θ' (or T') out of phase by 1/4 wavelength:

Warm air leads pressure ridge eastward

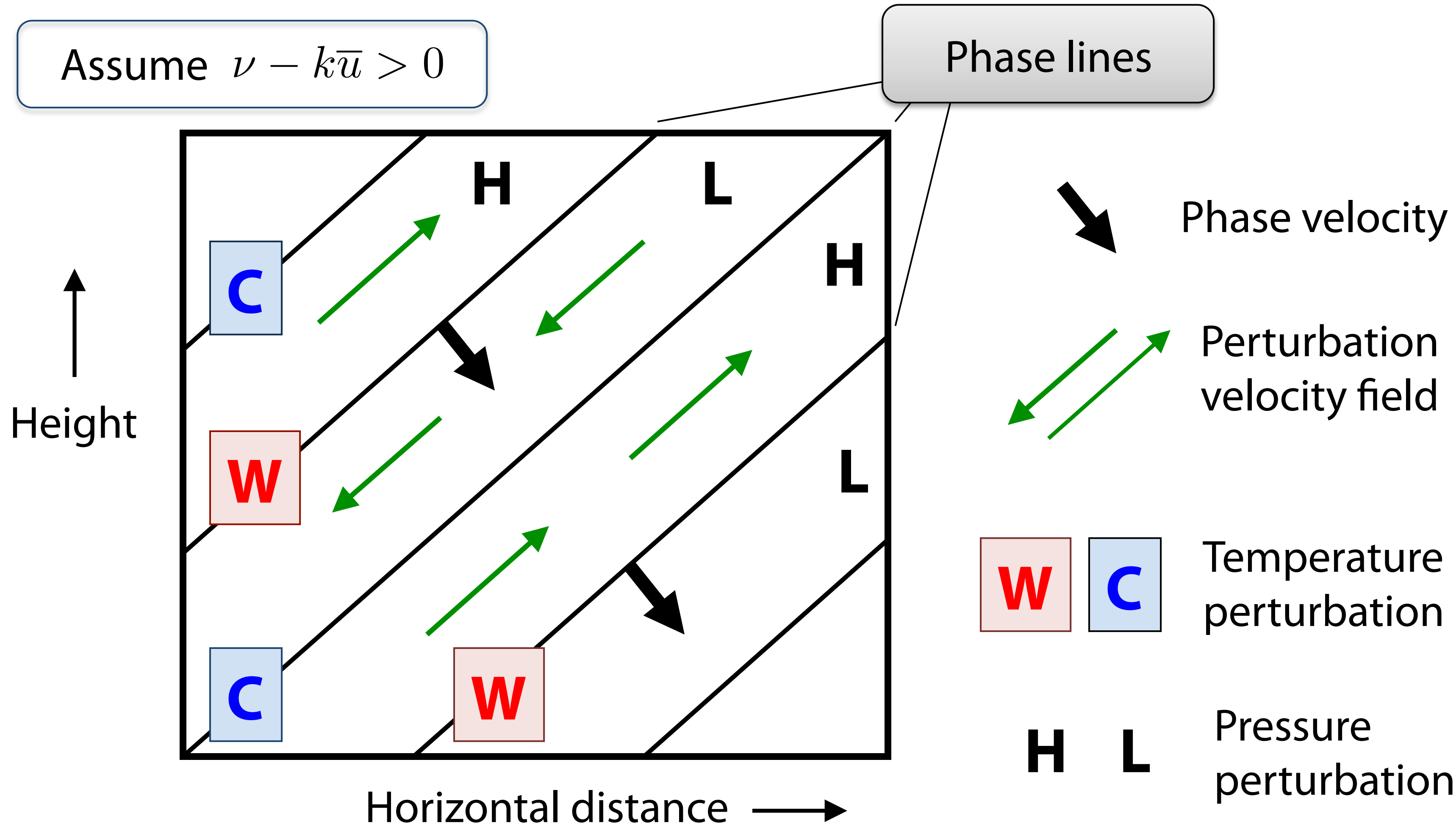
Perpendicular!

Transverse wave: $\mathbf{k} \cdot \mathbf{u} \equiv ku' + mw' = w' \left[k\left(-\frac{m}{k}\right) + m\right] = 0$

IGWs: Cross Section

If we let $k > 0$ and $m < 0$, then lines of constant phase $\phi = (kx + mz)$ tilt eastward with increasing height

Assume $\nu - k\overline{u} > 0$



IGWs: Phase / Group Velocity

The zonal and vertical **phase velocities** are

$$c_x = \frac{\nu}{k} = \overline{u} \pm \frac{\mathcal{N}}{\sqrt{k^2 + m^2}}$$

$$c_z = \frac{\nu}{m} = \frac{\overline{u}k}{m} \pm \frac{\mathcal{N}k}{m\sqrt{k^2 + m^2}}$$

The components of the **group velocity** are

$$c_{g,x} = \frac{\partial \nu}{\partial k} = \overline{u} \pm \frac{\mathcal{N}m^2}{(k^2 + m^2)^{3/2}}$$

$$c_{g,z} = \frac{\partial \nu}{\partial m} = \mp \frac{\mathcal{N}km}{(k^2 + m^2)^{3/2}}$$

Upper and lower signs chosen as above.

IGWs: Phase / Group Velocity

The **vertical** component of **group velocity** has a sign **opposite** to that of the **vertical phase speed.**

That is, **downward phase** propagation implies **upward energy** propagation

$$c_x = \frac{\nu}{k} = \overline{u} \pm \frac{\mathcal{N}}{\sqrt{k^2 + m^2}}$$

$$c_z = \frac{\nu}{m} = \frac{\overline{u}k}{m} \pm \frac{\mathcal{N}k}{m\sqrt{k^2 + m^2}}$$

$$c_{g,x} = \frac{\partial \nu}{\partial k} = \overline{u} \pm \frac{\mathcal{N}m^2}{(k^2 + m^2)^{3/2}}$$

$$c_{g,z} = \frac{\partial \nu}{\partial m} = \mp \frac{\mathcal{N}km}{(k^2 + m^2)^{3/2}}$$

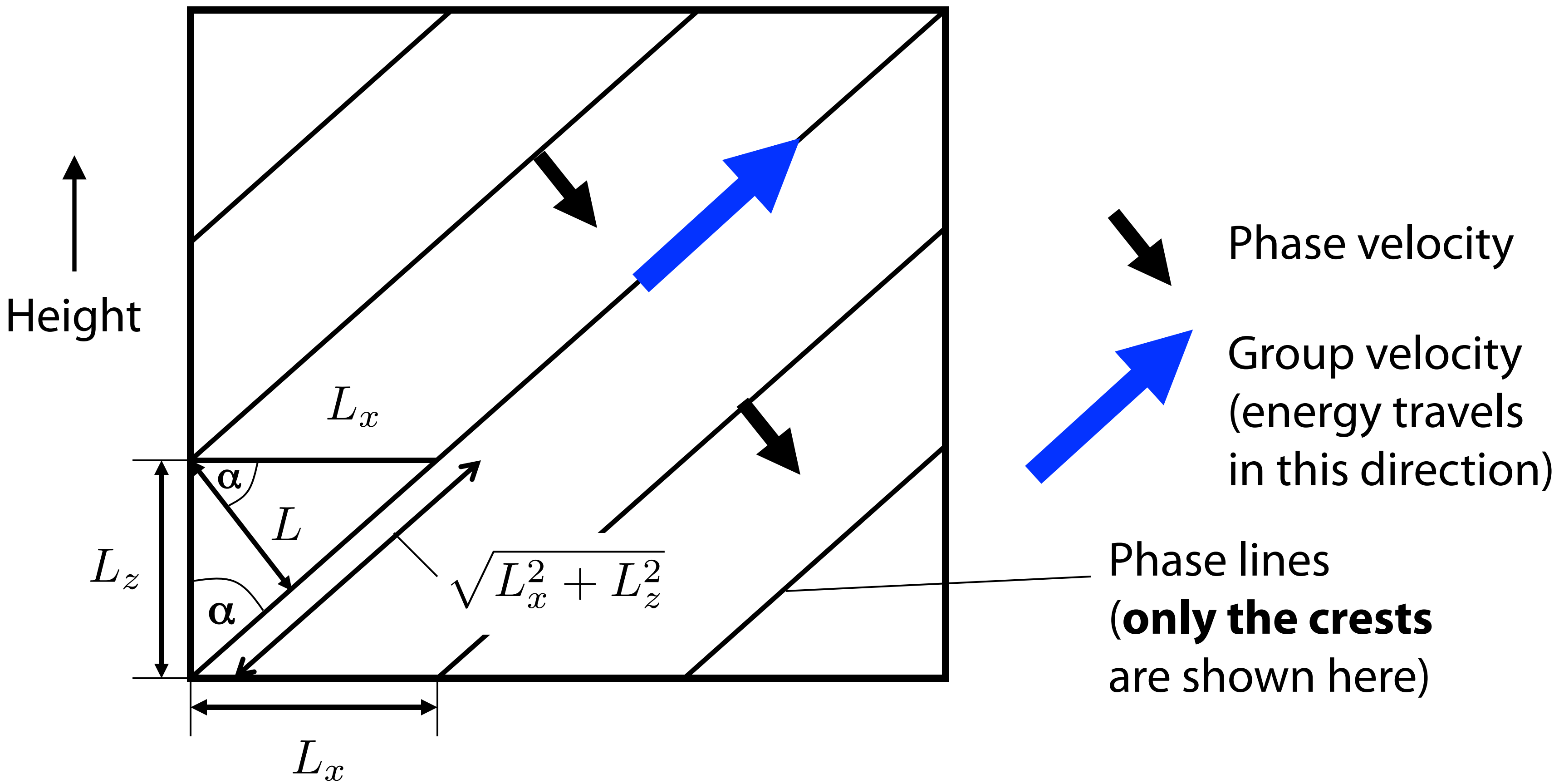
IGWs: Phase / Group Velocity

Horizontal phase and group velocities point in the same direction (no sign change)

Remarkable property of internal gravity waves: Group velocity travels perpendicular to the direction of the phase propagation

That is, in internal gravity waves **energy travels parallel to the crests and throughs** (in contrast to shallow water gravity waves)

IGWs: Phase / Group Velocity



Angle of phase lines to the local vertical:

$$\cos\alpha = \pm\frac{L}{L_x} = \pm\frac{L_z}{\sqrt{L_x^2+L_z^2}} = \pm\frac{\left(\frac{2\pi}{m}\right)}{\sqrt{\left(\frac{2\pi}{k}\right)^2+\left(\frac{2\pi}{m}\right)^2}} = \pm\frac{k}{\sqrt{k^2+m^2}} = \pm\frac{k}{\kappa}$$

IGWs: Frequency

Angle of phase lines to the local vertical:

$$\cos\alpha = \pm\frac{L}{L_x} = \pm\frac{L_z}{\sqrt{L_x^2 + L_z^2}} = \pm\frac{\left(\frac{2\pi}{m}\right)}{\sqrt{\left(\frac{2\pi}{k}\right)^2 + \left(\frac{2\pi}{m}\right)^2}} = \pm\frac{k}{\sqrt{k^2+m^2}} = \pm\frac{k}{\kappa}$$

Frequency of internal gravity waves:

$$\hat{\nu} = \nu - \overline{u}k = \pm\frac{\mathcal{N}k}{\sqrt{k^2+m^2}} = \pm\frac{\mathcal{N}k}{\kappa}$$

Intrinsic Frequency

$$\hat{\nu} \equiv \nu - \overline{u}k = \pm\mathcal{N}\cos\alpha$$

Consequence: gravity wave **intrinsic frequency** (relative to mean flow) must be less than buoyancy frequency

IGWs: Movies

Propagation of an internal gravity wave packet

https://www.youtube.com/watch?v=cDsNmnpq9_o

Observed waves in the atmosphere

http://www.youtube.com/watch?v=RpP62QSJM0g

Topographic Gravity Waves

Topographic gravity waves are stationary internal waves driven by topography (a lower boundary condition on w).

For stationary waves (ν=0) this means that the intrinsic frequency is $\hat{\nu} = -\overline{u}k < 0$ (westward relative to the mean flow)

Wavelength k is determined by the wavelength of the topography (flow follows topography near the ground)

For example, idealized mountain height profile:

$$h(x) = h_M \cos(kx)$$

Topographic Gravity Waves

Topographic gravity waves are stationary internal waves driven by topography (a lower boundary condition on w).

For stationary waves, the dispersion relationship for IGWs

$$\left(\cancel{\frac{\partial}{\partial t}} + \overline{u}\frac{\partial}{\partial x} \right)^2 \left(\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial z^2} \right) + \mathcal{N}^2 \frac{\partial^2 w'}{\partial x^2} = 0$$

simplifies to

$$\left(\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial z^2} \right) + \frac{\mathcal{N}^2}{\overline{u}^2} w' = 0$$

Topographic Gravity Waves

Seek stationary ($\nu = 0, \hat{\nu} = \overline{u}k$) wave solutions:

$$w' = \mathrm{Re}\left[\hat{w}\exp(i\phi)\right] = w_r\cos\phi - w_i\sin\phi$$

$$\hat{w} = w_r + iw_i, \quad m = m_r + im_i, \quad \phi = kx + mz$$

yields the dispersion relationship

$$\boxed{m^2 = \frac{\mathcal{N}^2}{\overline{u}^2} - k^2}$$

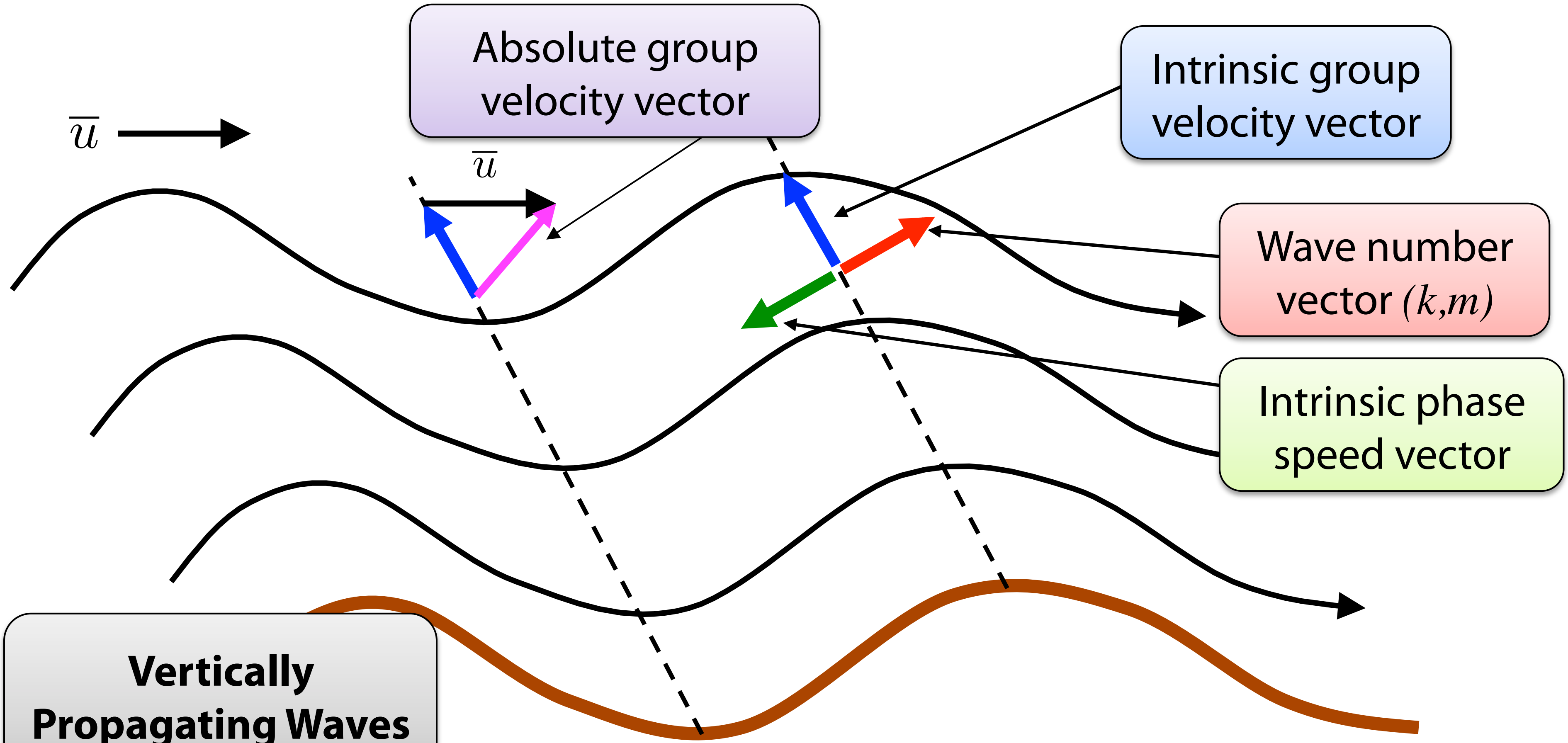
If $|\overline{u}| < \mathcal{N}/k$, $m^2 > 0$ (m must be real). Solutions have the form of **vertically propagating waves** (with constant amplitude):

$$\boxed{w' = \hat{w}\exp(i(kx + mz))}$$

Wide (broad) mountain $\Longleftrightarrow$ small k $\Longleftrightarrow$ m real

Ridges and trough lines slope with height, opposite to u

Upward flow of energy and momentum



Vertically Propagating Waves

Topographic Gravity Waves

If $|\overline{u}| > \mathcal{N}/k$, $m^2 < 0$ (m is purely imaginary). Solutions have the form of **vertically trapped waves** (that decay with height):

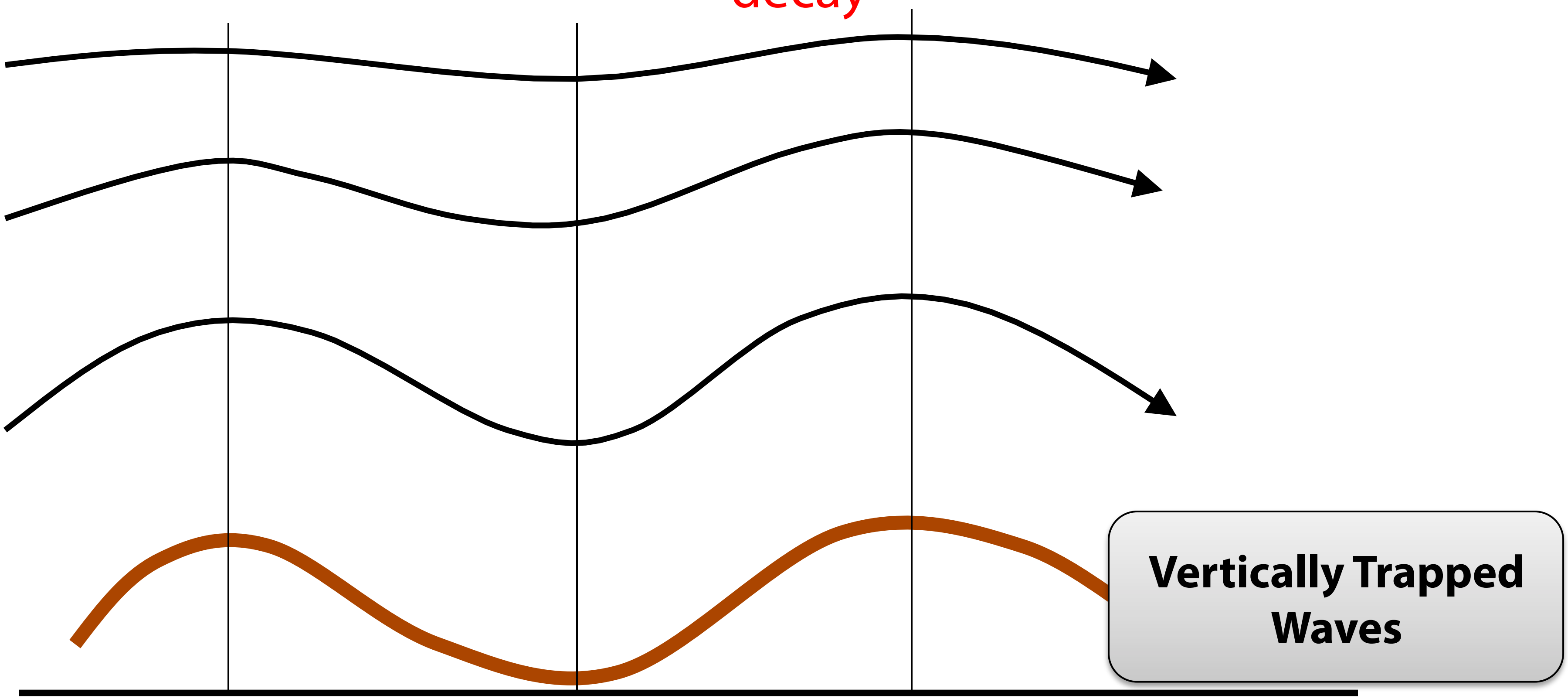
$$m^2 = \frac{\mathcal{N}^2}{\overline{u}^2} - k^2 \quad \Rightarrow \quad \boxed{w' = \hat{w}\exp(ikx)\exp(-m_i z)}$$

Narrow mountain $\Longleftrightarrow$ large k $\Longleftrightarrow$ m imaginary

Ridges and trough lines aligned with height

No tilt, no momentum transport

$$w' = \hat{w} \exp(ikx) \underbrace{\exp(-m_i z)}_{\text{decay}}$$



Topographic Gravity Waves

Summary

Vertical propagation of gravity waves is only possible when $|\overline{u}k|$ is **less than the buoyancy frequency** $\mathcal{N}$

Stable stratification, wide ridges and relatively weak zonal flow provide favorable conditions for vertically propagating gravity waves.

If energy is **transported upward**, phase must be **downward**.

More Realistic IGWs

Figure: Circulation is more complex in real gravity waves: For example, rotors, breaking waves, wind shear, varying $\mathcal{N}$

Lenticularis Cloud

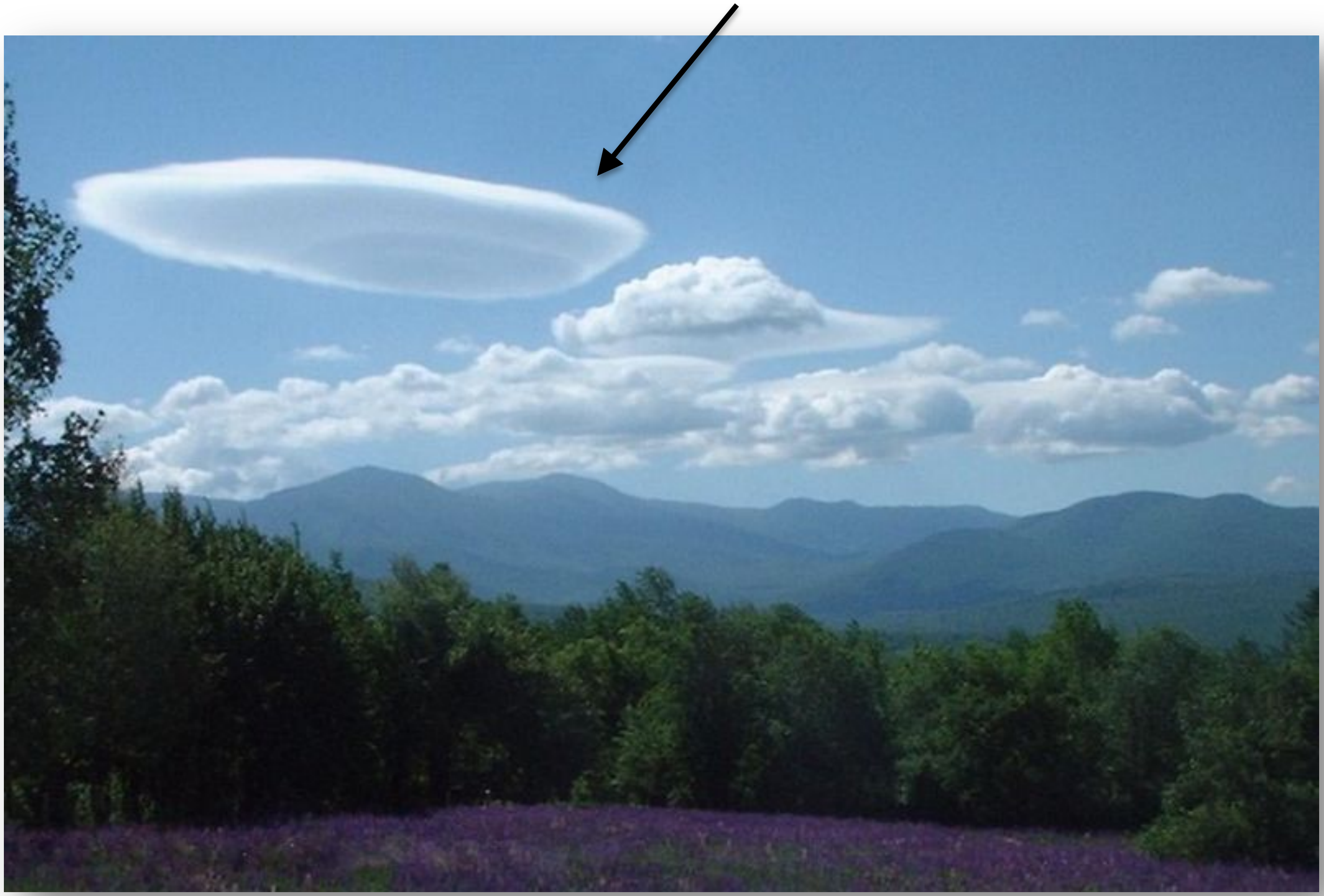


Figure: Lenticularis cloud forming on the lee side of the mountain

Movie: https://www.youtube.com/watch?v=YKAfKHSeWZc