Atmospheric Waves

Chapter 6

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Part 1: Wave-like Motion

Atmospheric Waves

The equations of motion contain many forms of wave-like solutions, true for the atmosphere and ocean. Waves are important since they **transport energy** and **mix the air** (especially when breaking).

Some are of interest depending on the problem: Rossby waves, internal gravity (buoyancy) waves, inertial waves, inertial-gravity waves, topographic waves, shallow water gravity waves.

Some are not of interest to meteorologists, for instance sound waves.

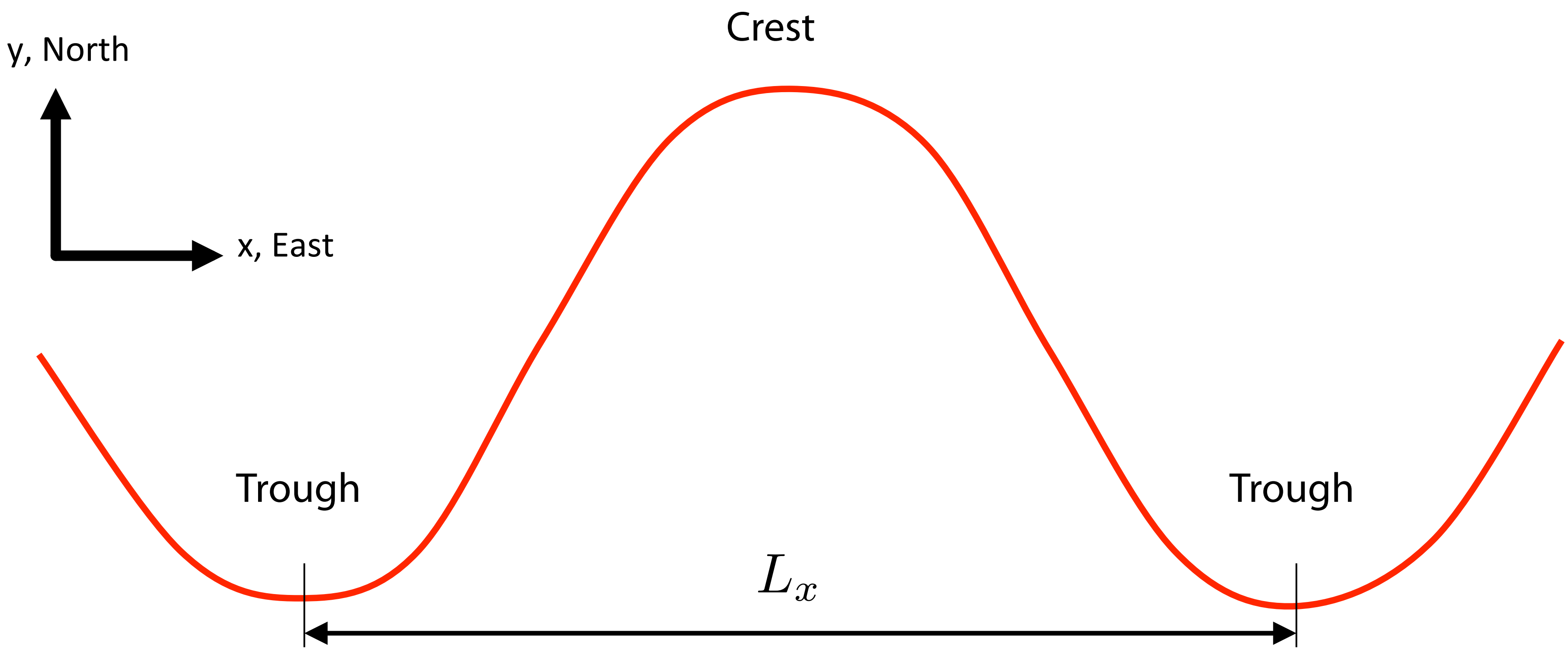
Atmospheric Waves

Large-scale mid-latitudinal waves (**Rossby waves**) are critical for weather forecasting and transport (see Atmospheric Waves).

Large-scale waves in the tropics (**Kelvin waves, mixed Rossby-gravity waves**) are also important, but of very different character.

Waves can be unstable. That is they start to grow, rather than just bounce back and forth.

Atmospheric Waves

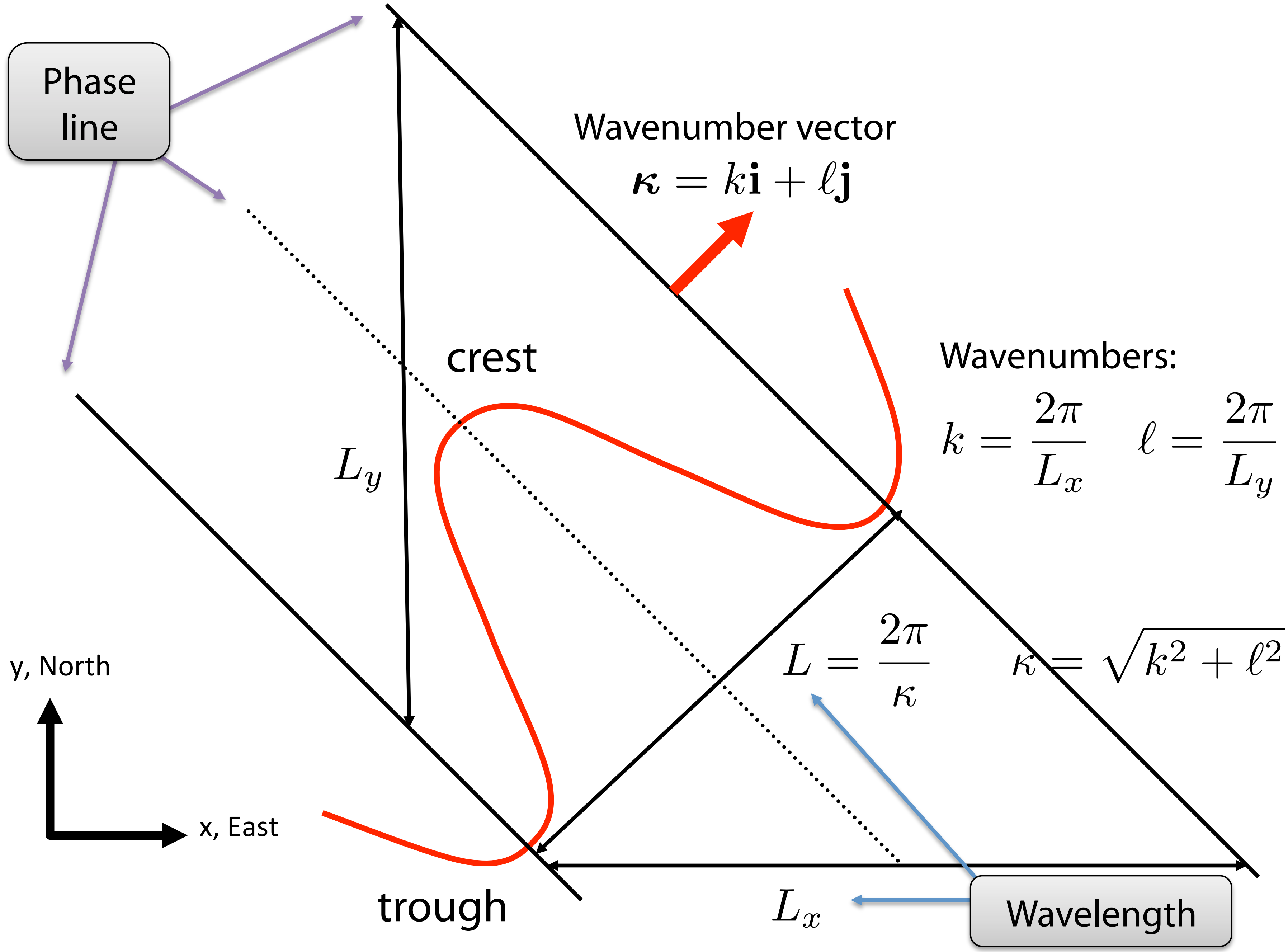


Definition: The wavelength L_x of a wave is the distance between neighboring troughs (or crests).

Definition: The wavenumber k of a wave is defined as

$$k \equiv \frac{2\pi}{L_x}$$

Atmospheric Wave Example

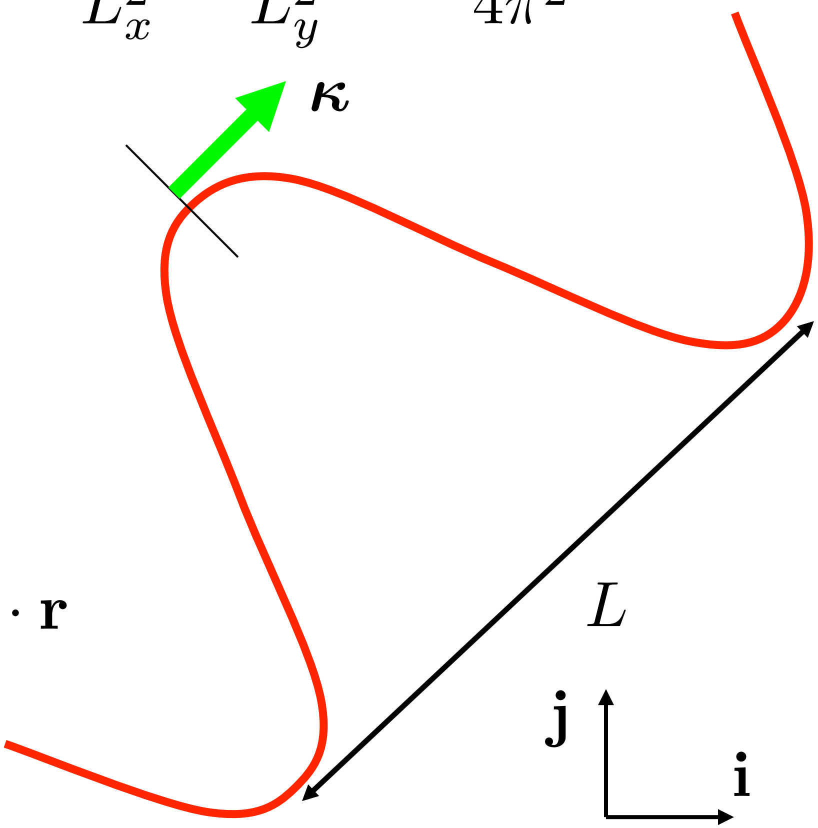

Phase line
Wavenumber vector
$\boldsymbol{\kappa} = k\mathbf{i} + \ell\mathbf{j}$
crest
Wavenumbers:
$k = \frac{2\pi}{L_x}$
$\ell = \frac{2\pi}{L_y}$
L_y
$L = \frac{2\pi}{\kappa}$
$\kappa = \sqrt{k^2 + \ell^2}$
y, North
x, East
trough
L_x
Wavelength

Waves in 2D

Wavelengths: $L_x = \frac{2\pi}{k} \quad L_y = \frac{2\pi}{\ell}$

From geometric considerations: $\frac{1}{L^2} = \frac{1}{L_x^2} + \frac{1}{L_y^2} = \frac{k^2 + \ell^2}{4\pi^2}$

$$\boxed{L = \frac{2\pi}{\kappa} \qquad \kappa = \sqrt{k^2 + \ell^2}}$$



Wavenumber vector: $\boldsymbol{\kappa} = k\mathbf{i} + \ell\mathbf{j}$

Constant phase lines: $kx + \ell y = \mathbf{k} \cdot \mathbf{r}$

Vector position: $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$

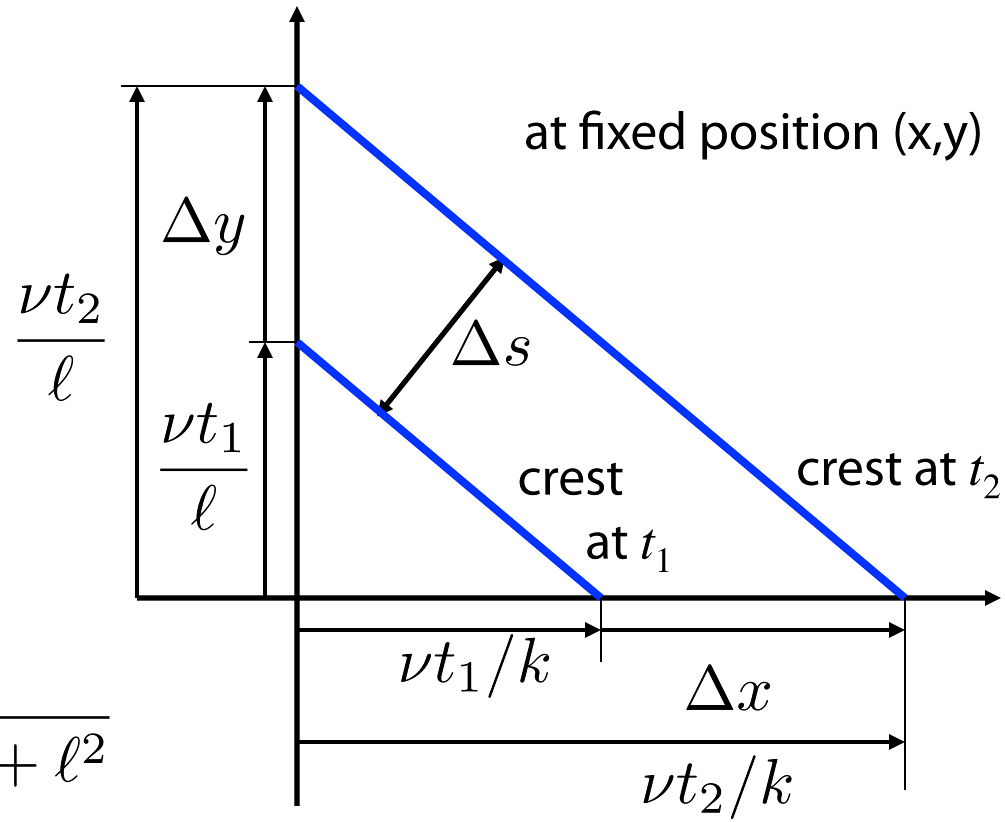
2D Traveling Waves

Wave frequency: $\nu = \frac{2\pi}{T}$ with T wave period

Geometric considerations provide:

$$\frac{1}{\Delta s^2} = \frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}$$

$$\Delta s = \frac{\nu \Delta t}{\kappa}$$



Phase speed:

$$c = \frac{\Delta s}{\Delta t} = \frac{\nu}{\kappa}$$

with wavenumber $\kappa = \sqrt{k^2 + \ell^2}$

2D Traveling Waves

The crests are translated over a distance:

$$\Delta x = \frac{\nu t_2}{k} - \frac{\nu t_1}{k} = \frac{\nu \Delta t}{k}$$

Propagation speed of the wave:

$$c_x = \frac{\Delta x}{\Delta t} = \frac{\nu}{k}$$ in x-direction

$$c_y = \frac{\Delta y}{\Delta t} = \frac{\nu}{\ell}$$ in y-direction

Propagation speed of the crest line: **phase speed**

$$c = \frac{\Delta s}{\Delta t} = \frac{\nu}{\kappa}$$

direction is parallel to wavenumber vector $\boldsymbol{\kappa}$

Phase speed c is less than either c_x or c_y

Dispersion Relationship

If the frequency ν is a function of the wavenumber components, so is the phase speed:

$$c(k,\ell) = \frac{\nu(k,\ell)}{\sqrt{k^2+\ell^2}}$$

Physically, this implies that various waves of a composite signal will all travel at different speeds.

As a result, there is distortion of the signal over time.

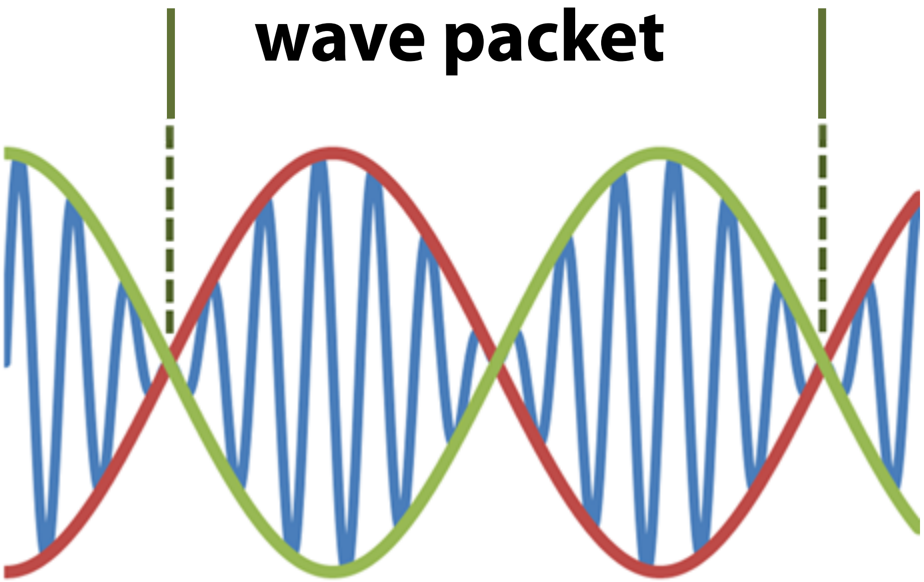
This phenomenon is called **wave dispersion.**

Definition: The **dispersion relationship** is then defined by $\nu(k,\ell)$

Wave Envelope

In general, a wave pattern consists of a series of superimposed waves, leading to destructive and constructive interference.

Therefore: Energy distribution is a property of a set of waves rather than a single wave.



Source: Wikipedia

Wave Envelope

A wave pattern is a succession of wave packets.

Within each packet (here 1D), the wave propagates at the phase speed

$$c = \frac{\nu}{k}$$

While the packet or envelope (and therefore the energy) travels at the **group velocity** (here in 1D)

Definition: The **group velocity** of a system is defined as

$$c_g = \frac{\partial \nu}{\partial k}$$

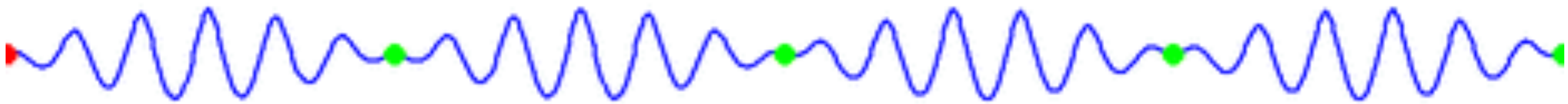
The **group velocity vector** (in 2D) is defined as

$$\mathbf{c}_g \equiv \nabla_{\mathbf{k}} \nu$$

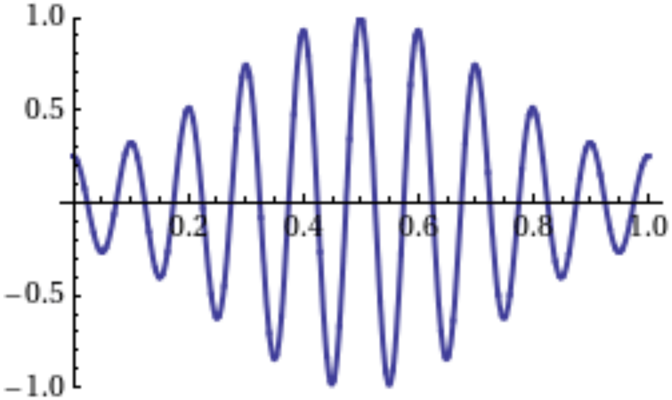
Components of group velocity vector:

$$c_{gx} = \frac{\partial \nu}{\partial k} \qquad c_{gy} = \frac{\partial \nu}{\partial \ell}$$

Group Velocity



Source: Wikipedia



Source: Wikipedia

Linear Perturbation Theory

Assume that each variable is equal to the mean state plus a perturbation from that mean:

$$\underbrace{u}_{\text{Variable}} = \underbrace{\overline{u}}_{\text{Mean}} + \underbrace{u'}_{\text{Difference (perturbation)}}$$

This is a very general approach, and can always be done.

Linear Perturbation Theory

Impose assumptions and constraints on the mean and perturbations

Assumptions need to be physically sensible and justified

For example:

$$u = \overline{u} + u'$$

Variable

Mean (independent of time and longitude)

Perturbation: Perturbations are "small", and so products of perturbations are very small.

Linear Perturbation Theory

With these assumptions **non-linear terms** (like the one below) become **linear**:

$$u\frac{\partial u}{\partial x} = (\overline{u} + u')\frac{\partial}{\partial x}(\overline{u} + u')$$

$$= \cancel{\overline{u}\frac{\partial \overline{u}}{\partial x}} + \cancel{u'\frac{\partial \overline{u}}{\partial x}} + \overline{u}\frac{\partial u'}{\partial x} + \cancel{u'\frac{\partial u'}{\partial x}}$$

These terms are zero since the mean is independent of x.

Terms with products of perturbations are very small and will be ignored.

$$\boxed{u\frac{\partial u}{\partial x} \approx \overline{u}\frac{\partial u'}{\partial x}}$$

Linearization

Assume that the mean state satisfies the equations of motion

For example: u-momentum equation becomes (here with constant density, frictionless):

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv$$

Plug in: $u = \overline{u} + u' \quad v = v' \quad p = \overline{p} + p'$

Linearize: $$\frac{\partial u'}{\partial t} + \overline{u}\frac{\partial u'}{\partial x} + v'\frac{\partial \overline{u}}{\partial y} = -\frac{1}{\rho}\frac{\partial p'}{\partial x} + fv'$$

1D Wave Solutions

Assume we have derived a set of linearized equations with constant coefficients.

Look for simple wave-like solutions:

$$u'(x,t) = u_0 \cos(kx - \nu t) \quad \text{with} \quad k = \frac{2\pi}{L_x} \quad \nu = \frac{2\pi}{T}$$

However, this form of the wave fixes the "phase" over the wave. That is at $x=0$ and $t=0$ all solutions must have $u' = u_0$.

More general solution: Use complex numbers

$$\boxed{i = \sqrt{-1}}$$

$$u'(x,t) = \mathrm{Re}\left[u_0 \exp(ikx - i\nu t)\right] \qquad u_0 = \mathrm{Re}(u_0) + i\mathrm{Im}(u_0)$$

2D Wave Solutions

2D waves that propagate horizontally in the x and y direction with constant wave amplitude u_0

$$u'(x, y, t) = \mathrm{Re}\left[u_0 \exp(i(kx + \ell y - \nu t))\right]$$

Real part of

Wave amplitude, may be complex

Recall: $$\exp(i\phi) = \cos\phi + i\sin\phi$$

$$\Rightarrow \quad \mathrm{Re}\left[u_0 \exp(i\phi)\right] = \mathrm{Re}(u_0)\cos\phi - \mathrm{Im}(u_0)\sin\phi$$

Note: Only the real part Re[ ] has physical meaning!

3D Wave Solutions

In 3D search for general wave solutions of the form:

$$u'(x, y, z, t) = \text{Re}\left[u_0 \exp(i(kx + \ell y + mz - \nu t))\right]$$

Wave amplitude, may be complex

Such a wave propagates in all three dimensions.

Alternatively: Horizontally propagating waves with varying amplitude $u_0(z)$ in the vertical direction:

$$u'(x, y, z, t) = \text{Re}\left[u_0(z) \exp(i(kx + \ell y - \nu t))\right]$$