Quasi-Geostrophic Theory
Chapter 4

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Part 5: Quasi-Geostrophic Potential Vorticity

QG Geopotential Tendency

QG Geopotential Tendency Eq'n (Adiabatic)

$$\underbrace{\left[\nabla^2 + \frac{\partial}{\partial p}\left(\frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\right)\right]\chi}_{\text{Local geopotential tendency}} = \underbrace{-f_0 \mathbf{u}_g \cdot \nabla \left(\frac{1}{f_0}\nabla^2\Phi + f\right)}_{\text{Proportional to absolute vorticity advection}} - \frac{\partial}{\partial p}\underbrace{\left[-\frac{f_0^2}{\sigma}\mathbf{u}_g \cdot \nabla\left(-\frac{\partial \Phi}{\partial p}\right)\right]}_{\text{Proportional to (differential) thickness (temperature) advection}}$$

Local geopotential tendency

Proportional to absolute vorticity advection

Proportional to (differential) thickness (temperature) advection

Only the initial distribution of Φ needs to be known!

QG Potential Vorticity

Starting from here:

$$\left[\nabla^2 + \frac{\partial}{\partial p}\left(\frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\right)\right]\chi = -f_0\mathbf{u}_g \cdot \nabla\left(\frac{1}{f_0}\nabla^2\Phi + f\right) - \frac{\partial}{\partial p}\left[-\frac{f_0^2}{\sigma}\mathbf{u}_g \cdot \nabla\left(-\frac{\partial\Phi}{\partial p}\right)\right]$$

Observe that by chain rule the last term expands as:

$$-\frac{\partial}{\partial p}\left[-\frac{f_0^2}{\sigma}\mathbf{u}_g \cdot \nabla\left(-\frac{\partial\Phi}{\partial p}\right)\right] = \underbrace{\frac{f_0^2}{\sigma}\frac{\partial\mathbf{u}_g}{\partial p} \cdot \nabla\left(-\frac{\partial\Phi}{\partial p}\right)}_{\text{Look closely at this term}} + \frac{f_0^2}{\sigma}\mathbf{u}_g \cdot \nabla\frac{\partial}{\partial p}\left(\frac{f_0^2}{\sigma}\frac{\partial\Phi}{\partial p}\right)$$

QG Potential Vorticity

$$-\frac{\partial}{\partial p}\left[-\frac{f_0^2}{\sigma}\mathbf{u}_g \cdot \nabla\left(-\frac{\partial \Phi}{\partial p}\right)\right] = \frac{f_0^2}{\sigma}\frac{\partial \mathbf{u}_g}{\partial p} \cdot \nabla\left(-\frac{\partial \Phi}{\partial p}\right) + \mathbf{u}_g \cdot \nabla \frac{\partial}{\partial p}\left(\frac{f_0^2}{\sigma}\frac{\partial \Phi}{\partial p}\right)$$

Use thermal wind relationship (vector form):

$$f_0 \frac{\partial \mathbf{u}_g}{\partial p} = \mathbf{k} \times \nabla\left(\frac{\partial \Phi}{\partial p}\right)$$

Observe: $\frac{\partial \mathbf{u}_g}{\partial p}$ is perpendicular to $\nabla\left(\frac{\partial \Phi}{\partial p}\right)$

$$-\frac{\partial}{\partial p}\left[-\frac{f_0^2}{\sigma}\mathbf{u}_g \cdot \nabla\left(-\frac{\partial \Phi}{\partial p}\right)\right] = \cancel{\frac{f_0^2}{\sigma}\frac{\partial \mathbf{u}_g}{\partial p} \cdot \nabla\left(-\frac{\partial \Phi}{\partial p}\right)} + \mathbf{u}_g \cdot \nabla \frac{\partial}{\partial p}\left(\frac{f_0^2}{\sigma}\frac{\partial \Phi}{\partial p}\right)$$

QG Potential Vorticity

$$\left[\nabla^2 + \frac{\partial}{\partial p}\left(\frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\right)\right]\chi = -f_0\mathbf{u}_g\cdot\nabla\left(\frac{1}{f_0}\nabla^2\Phi + f\right) - \frac{\partial}{\partial p}\left[-\frac{f_0^2}{\sigma}\mathbf{u}_g\cdot\nabla\left(-\frac{\partial\Phi}{\partial p}\right)\right]$$

$$-\frac{\partial}{\partial p}\left[-\frac{f_0^2}{\sigma}\mathbf{u}_g\cdot\nabla\left(-\frac{\partial\Phi}{\partial p}\right)\right] = -\mathbf{u}_g\cdot\nabla\frac{\partial}{\partial p}\left(\frac{f_0^2}{\sigma}\frac{\partial\Phi}{\partial p}\right)$$

$$\frac{\partial}{\partial t}\left[\frac{1}{f_0}\nabla^2\Phi + f + \frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial\Phi}{\partial p}\right)\right] + \mathbf{u}_g\cdot\nabla\left[\frac{1}{f_0}\nabla^2\Phi + f + \frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial\Phi}{\partial p}\right)\right] = 0$$

QG Potential Vorticity

$$\frac{\partial}{\partial t}\left[\frac{1}{f_0}\nabla^2\Phi + f + \frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial}{\partial p}\right)\right] + \mathbf{u}_g \cdot \nabla\left[\frac{1}{f_0}\nabla^2\Phi + f + \frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial\Phi}{\partial p}\right)\right] = 0$$

Definition: The **quasi-geostrophic potential vorticity** is defined as

$$q = \frac{1}{f_0}\nabla^2\Phi + f + \frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial\Phi}{\partial p}\right)$$

$$\frac{D_g q}{Dt} = 0$$

q is conserved following geostrophic motion.

QG Potential Vorticity Equation

PV Comparison

Units

Barotropic PV

$$PV = \frac{\zeta_g + f}{h}$$

$\mathrm{m}^{-1}\mathrm{s}^{-1}$

Quasi-Geostrophic PV

$$q = \frac{1}{f_0}\nabla^2\Phi + f + \frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial\Phi}{\partial p}\right)$$

s^{-1}

Knowing PV is powerful: By using inversion of the PV, one can determine Φ, and therefore $\mathbf{u}_g$ and T can be deduced (with given boundary conditions).

QG Potential Vorticity

Quasi-Geostrophic PV

$$q = \underbrace{\frac{1}{f_0}\nabla^2\Phi + f}_{\text{Absolute vorticity}} + \underbrace{\frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial\Phi}{\partial p}\right)}_{\text{Vertical stretching (change in thickness with height)}}$$

Barotropic / Baroclinic

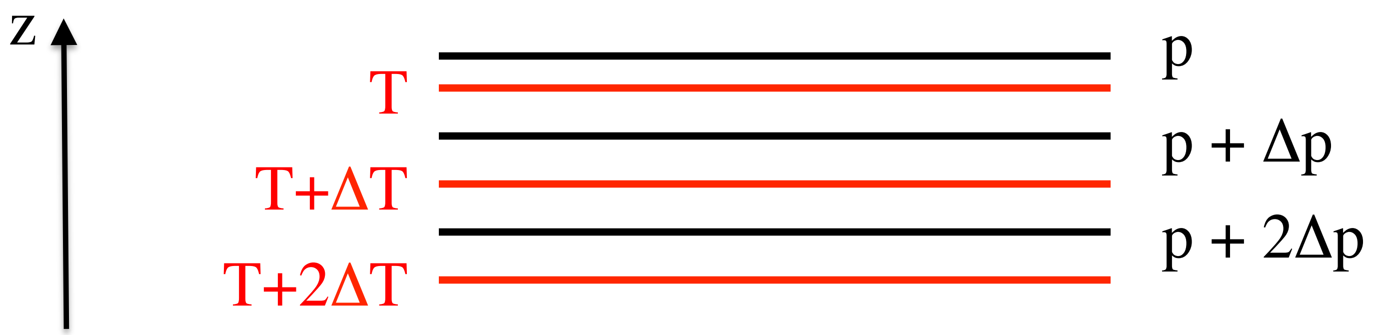
Definition: In a **barotropic fluid** density depends only on pressure.

By the ideal gas law, this implies that surfaces of constant density are surfaces of constant pressure are surfaces of constant temperature.

Definition: In a **baroclinic fluid** density depends on pressure and temperature.

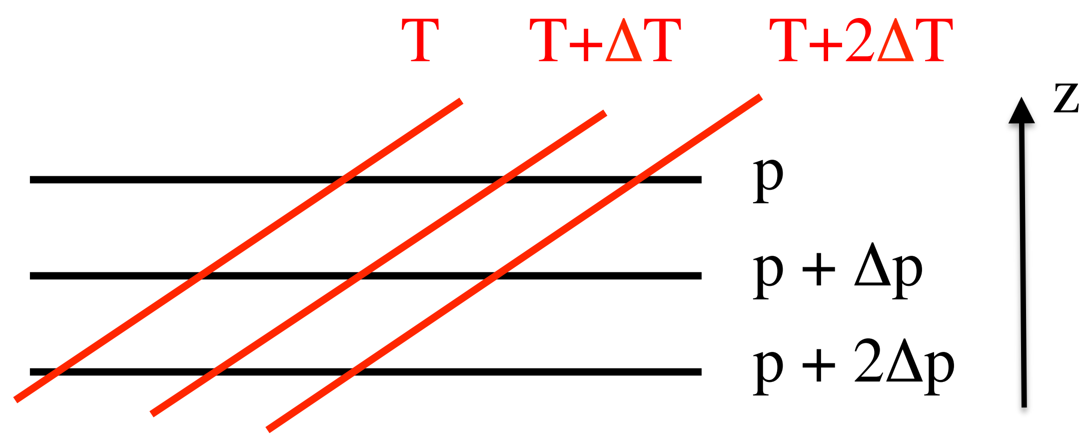
Barotropic / Baroclinic

Barotropic atmosphere:



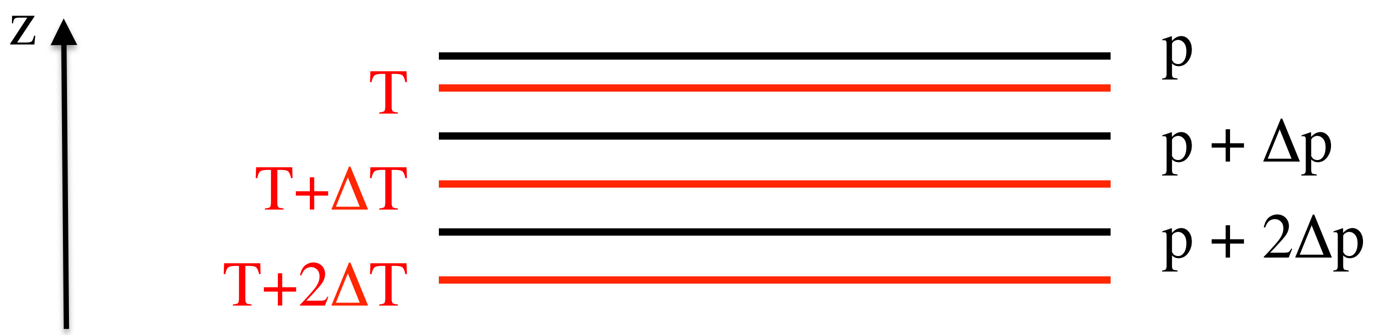
Baroclinic atmosphere:

A baroclinic fluid has energy that can be converted into motion.



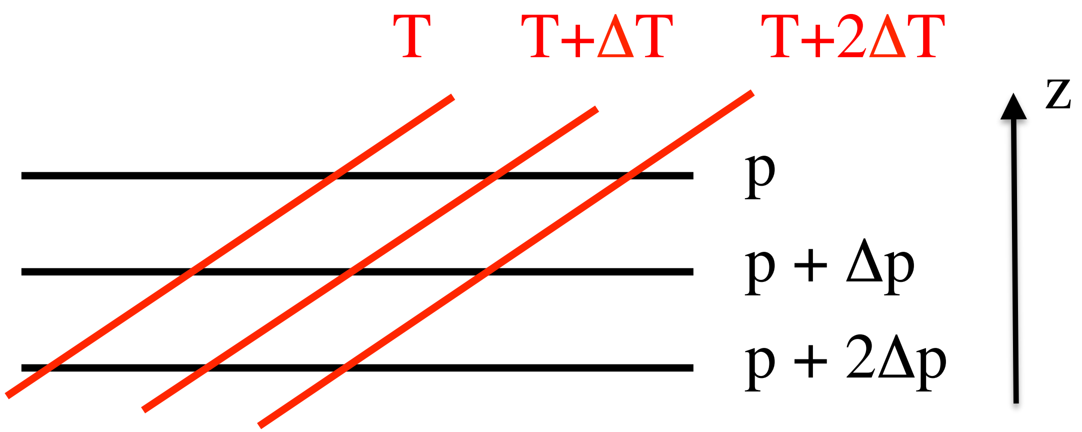
Barotropic / Baroclinic

Barotropic atmosphere:



Baroclinic atmosphere:

In particular, **diabatic heating** drives the development of temperature gradients.



QG Omega Equation

Recall: Definition of vertical pressure velocity $\omega \equiv \frac{Dp}{Dt}$

Question: Can we use the QG system to understand vertical motion? Stretching / vorticity generation? Clouds / precipitation?

QG Equations

Geostrophic Wind

$$\mathbf{u}_g = \frac{1}{f_0}\mathbf{k} \times \nabla\Phi$$

Momentum Equation

$$\frac{D_g\mathbf{u}_g}{Dt} = -f_0\mathbf{k} \times \mathbf{u}_a - \beta y\mathbf{k} \times \mathbf{u}_g$$

Continuity Equation

$$\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} + \frac{\partial \omega}{\partial p} = 0$$

Thermodynamic Equation

$$\left(\frac{\partial}{\partial t} + \mathbf{u}_g \cdot \nabla\right)\left(-\frac{\partial\Phi}{\partial p}\right) - \sigma\omega = \frac{\kappa J}{p}$$

Auxiliary Equations

$$\kappa \equiv \frac{R_d}{c_p} \qquad \sigma \equiv -\frac{R_d T_0}{p}\frac{\ln\theta_0}{p}$$

$\Phi, \mathbf{u}_g, \mathbf{u}_a, \omega$ are independent variables, form a complete set if heating rate J is known

QG Omega Equation

Step 1: Apply the horizontal Laplacian operator to the QG thermodynamic equation

$$\left(\frac{\partial}{\partial t} + \mathbf{u}_g \cdot \nabla\right)\left(-\frac{\partial \Phi}{\partial p}\right) - \sigma\omega = \frac{\kappa J}{p}$$

$$\Downarrow$$

$$\nabla^2 \frac{\partial \chi}{\partial p} = -\nabla^2 \left(\mathbf{u}_g \cdot \nabla \frac{\partial \Phi}{\partial p}\right) - \sigma \nabla^2 \omega - \frac{\kappa}{p} \nabla^2 J$$

QG Omega Equation

Step 2: Differentiate the geopotential height tendency equation with respect to pressure

$$\frac{1}{f_0}\nabla^2\chi = -\mathbf{u}_g \cdot \nabla \left(\frac{1}{f_0}\nabla^2\Phi + f \right) + f_0 \frac{\partial \omega}{\partial p}$$

$$\Downarrow$$

$$\nabla^2 \frac{\partial \chi}{\partial p} = -f_0 \frac{\partial}{\partial p} \left[\mathbf{u}_g \cdot \nabla \left(\frac{1}{f_0}\nabla^2\Phi + f \right) \right] + f_0^2 \frac{\partial^2 \omega}{\partial p^2}$$

QG Omega Equation

Step 3: Subtract the equations obtained from Step 1 and 2 to eliminate χ

$$\nabla^2 \frac{\partial \chi}{\partial p} = -\nabla^2 \left(\mathbf{u}_g \cdot \nabla \frac{\partial \Phi}{\partial p} \right) - \sigma \nabla^2 \omega - \frac{\kappa}{p} \nabla^2 J$$

$$\nabla^2 \frac{\partial \chi}{\partial p} = -f_0 \frac{\partial}{\partial p} \left[\mathbf{u}_g \cdot \nabla \left(\frac{1}{f_0} \nabla^2 \Phi + f \right) \right] + f_0^2 \frac{\partial^2 \omega}{\partial p^2}$$

$$\left(\nabla^2 + \frac{f_0^2}{\sigma} \frac{\partial^2}{\partial p^2} \right) \omega = \frac{f_0}{\sigma} \frac{\partial}{\partial p} \left[\mathbf{u}_g \cdot \nabla \left(\frac{1}{f_0} \nabla^2 \Phi + f \right) \right] + \frac{1}{\sigma} \nabla^2 \left[\mathbf{u}_g \cdot \nabla \left(-\frac{\partial \Phi}{\partial p} \right) \right] - \frac{\kappa}{\sigma p} \nabla^2 J$$

QG Omega Equation

Step 4: Expand terms on right-hand-side using chain rule, observing that 2 of the 4 terms cancel:

$$\left(\nabla^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega = \frac{f_0}{\sigma}\frac{\partial}{\partial p}\left[\mathbf{u}_g \cdot \nabla \left(\frac{1}{f_0}\nabla^2\Phi + f\right)\right] + \frac{1}{\sigma}\nabla^2\left[\mathbf{u}_g \cdot \nabla\left(-\frac{\partial \Phi}{\partial p}\right)\right] - \frac{\kappa}{\sigma p}\nabla^2 J$$

Adiabatic (J = 0)

Quasi-Geostrophic Omega Equation

$$\left[\nabla^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right]\omega \approx \frac{-2f_0}{\sigma}\left[-\frac{\partial \mathbf{u}_g}{\partial p}\cdot\nabla\left(\frac{1}{f_0}\nabla^2\Phi + f\right)\right]$$

QG Omega Equation

Quasi-Geostrophic Omega Equation

$$\left[\nabla^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right]\omega \approx \frac{-2f_0}{\sigma}\left[-\frac{\partial \mathbf{u}_g}{\partial p} \cdot \nabla \left(\frac{1}{f_0}\nabla^2\Phi + f\right)\right]$$

Generation of vertical velocity is governed by **advection of absolute vorticity** by the **thermal wind**.

QG Omega

Although small, vertical velocity is in many ways the key to weather and climate. It's important to waves growing and decaying. It governs how far the atmosphere is away from "balance."

QG Omega Equation

Quasi-Geostrophic Omega Equation

$$\underbrace{\left[\nabla^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right]}\omega \approx \frac{-2f_0}{\sigma}\left[-\frac{\partial \mathbf{u}_g}{\partial p}\cdot\nabla\left(\frac{1}{f_0}\nabla^2\Phi + f\right)\right]$$

For large scale atmospheric waves, this term is essentially a negative sign.

$$\omega = \omega_0 \sin\left(\frac{\pi p}{p_0}\right)\sin(kx)\sin(\ell y)$$

$$\left[\nabla^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right]\omega \approx -\left[k^2 + \ell^2 + \frac{1}{\sigma}\left(\frac{f_0\pi}{p_0}\right)^2\right]\omega \approx c\cdot w$$

Upward (downward) motion is forced if RHS of omega equation is positive (negative).

QG Omega Equation

The sign of w is proportional to the advection of absolute vorticity by the thermal wind

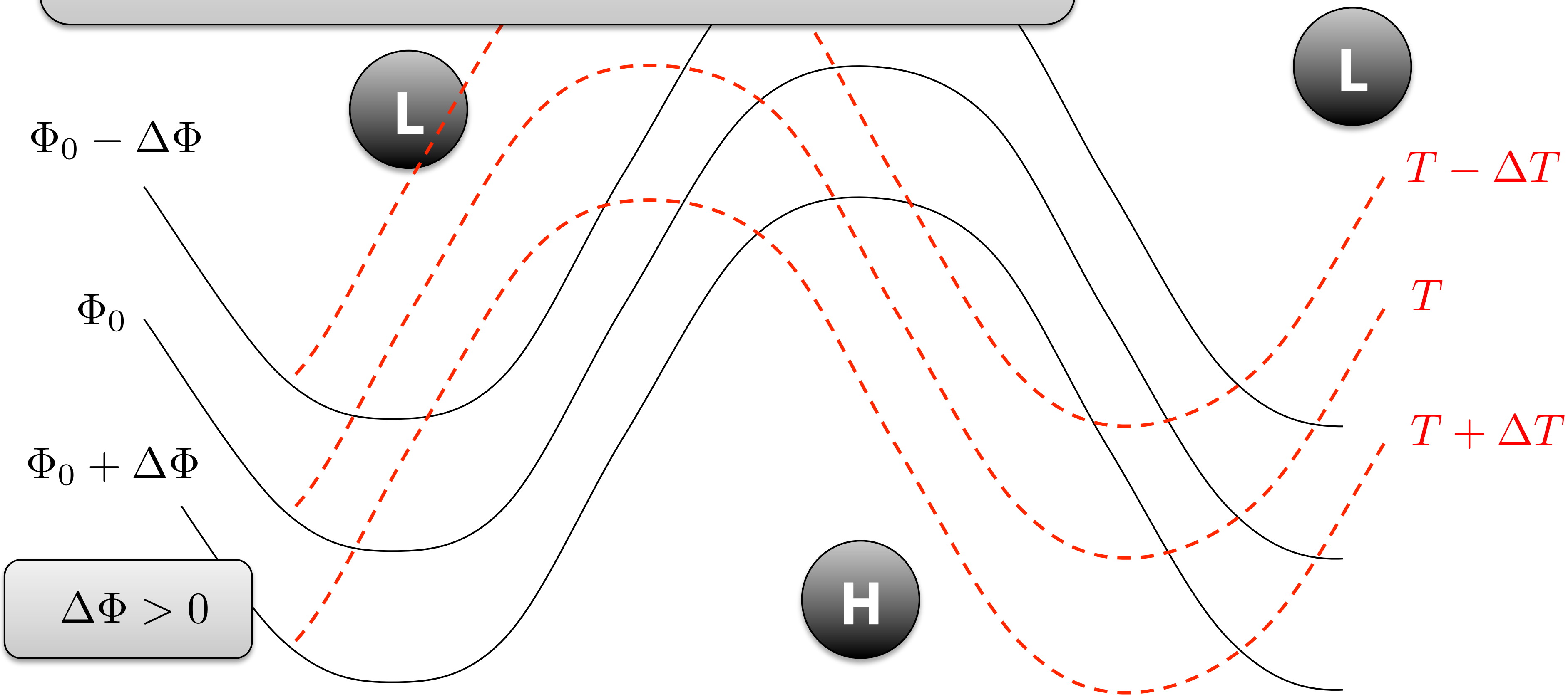
$$w \propto -\frac{\partial \mathbf{u_g}}{\partial z} \cdot \nabla(\zeta_g + f)$$

Note the change of vertical coordinate from p to z.

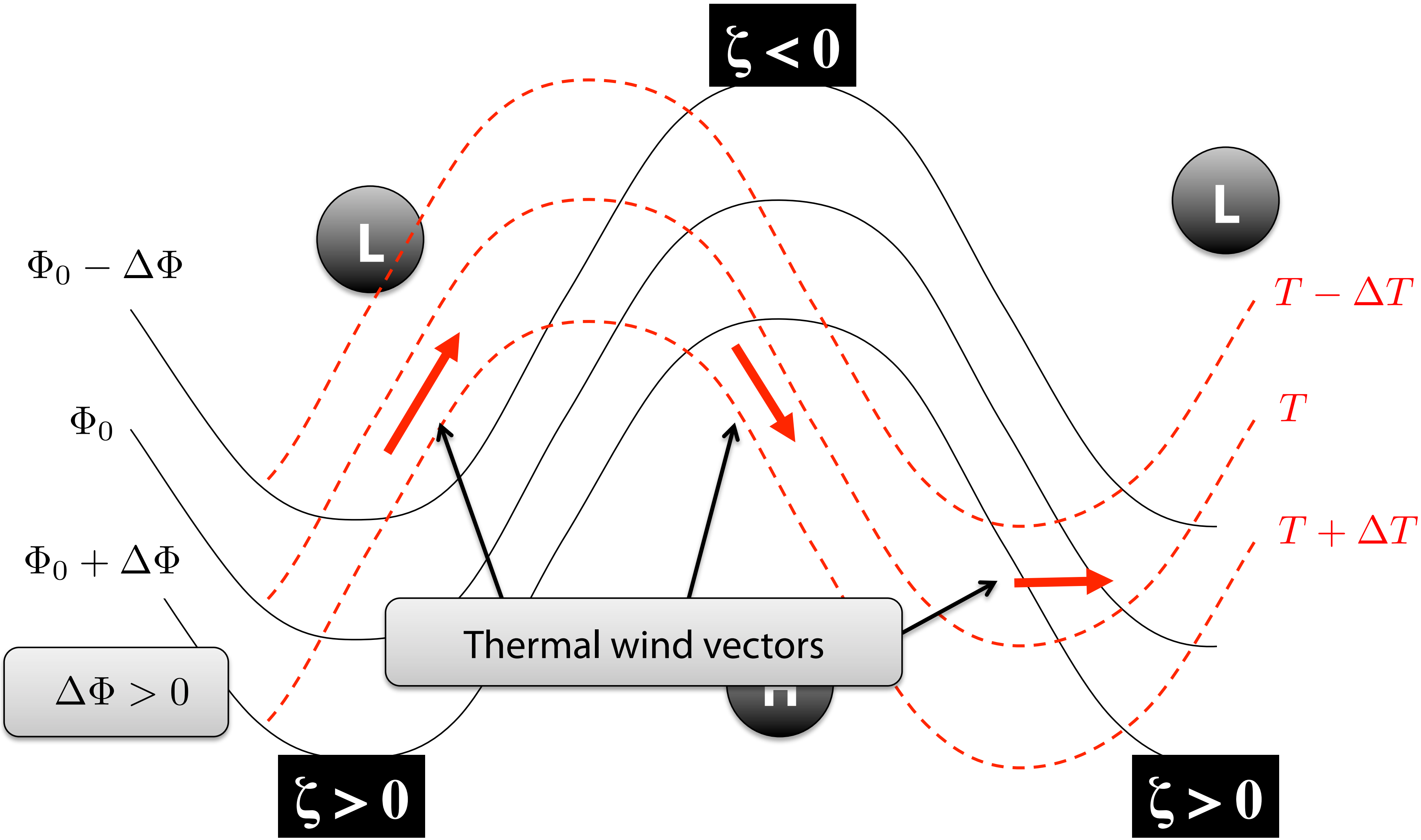
$$-\frac{\partial \mathbf{u_g}}{\partial p} \propto \frac{\partial \mathbf{u}_g}{\partial z} \propto \mathbf{u}_T$$

Baroclinic Instability

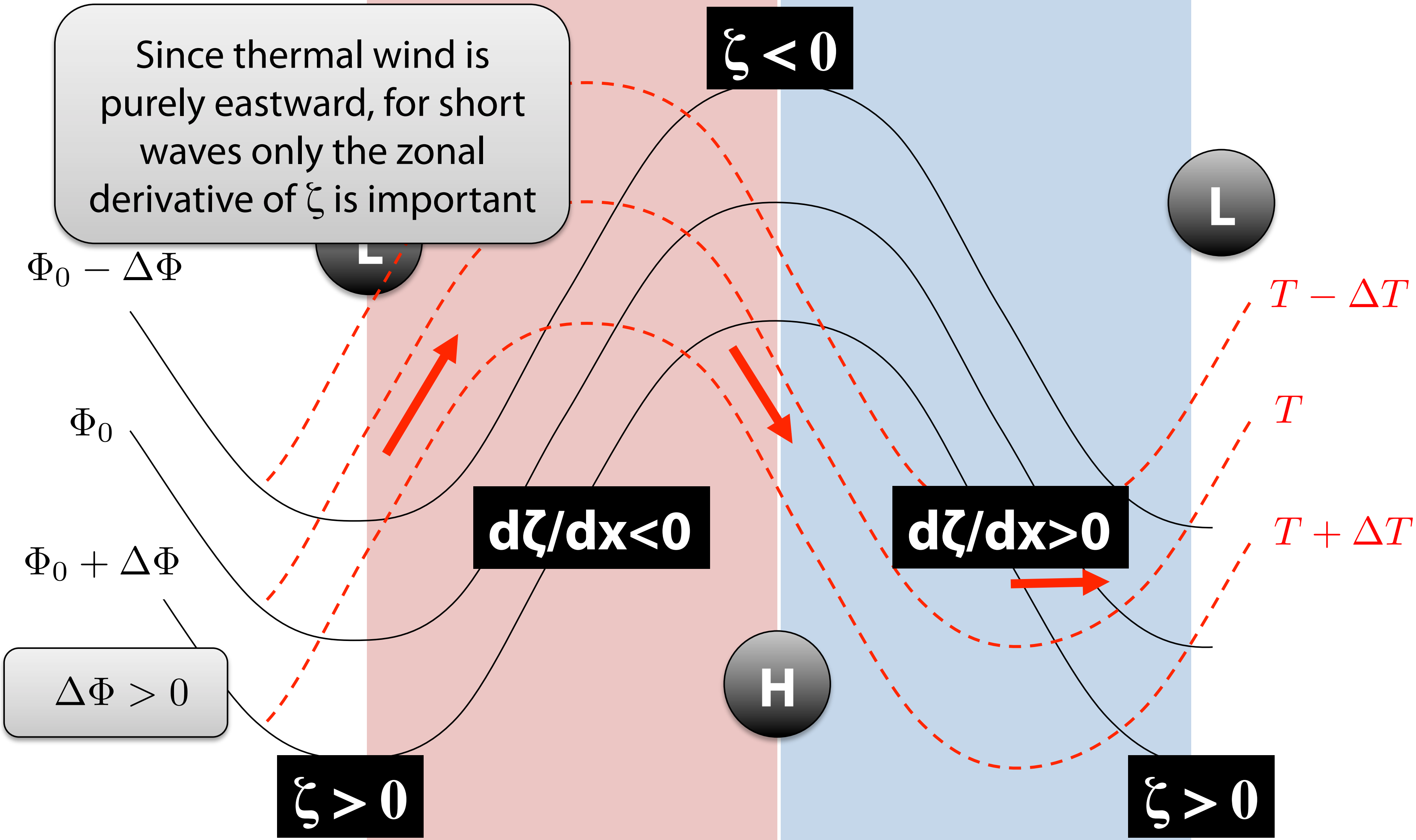
Due to westward tilt of system with height, isotherms are offset westward of geopotential contours.



Baroclinic Instability

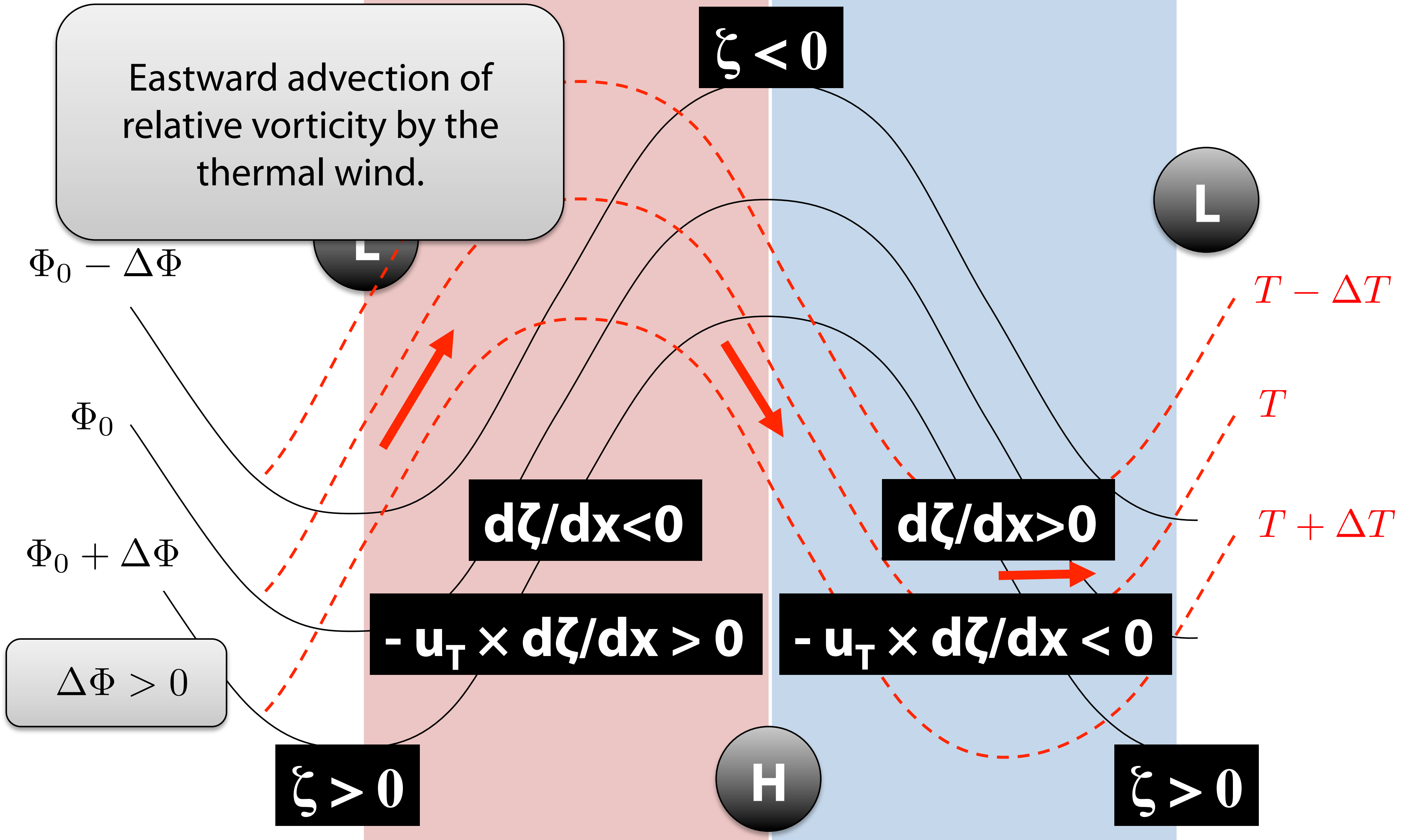


Baroclinic Instability



Baroclinic Instability

Eastward advection of relative vorticity by the thermal wind.



Baroclinic Instability

Rising (sinking) motion appears to the east (west) of the low.