Quasi-Geostrophic Theory
Chapter 4

Paul A. Ullrich
paullrich@ucdavis.edu

Part 4: The Geopotential Tendency Equation

Vorticity & Geopotential

QG Vorticity Equation

$$\frac{\partial \zeta_g}{\partial t} = f_0 \frac{\partial \omega}{\partial p} \boxed{- \mathbf{u}_g \cdot \nabla \zeta_g - v_g \beta}$$

Advection Terms

Geostrophic Wind

$$u_g = -\frac{1}{f_0}\frac{\partial \Phi}{\partial y} \qquad v_g = \frac{1}{f_0}\frac{\partial \Phi}{\partial x}$$

Geostrophic Vorticity

Relates geostrophic vorticity and geopotential

$$\zeta_g = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = \frac{1}{f_0}\left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}\right) = \frac{1}{f_0}\nabla^2 \Phi$$

QG Geopotential Tendency

Starting from here:

$$\frac{D_g \zeta_g}{Dt} = f_0 \frac{\partial \omega}{\partial p} - \frac{\partial f}{\partial y} v_g = f_0 \frac{\partial \omega}{\partial p} - \beta v_g$$

$$\zeta_g = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = \frac{1}{f_0} \nabla^2 \Phi$$

$$\frac{D_g}{Dt} \left(\frac{1}{f_0} \nabla^2 \Phi \right) = f_0 \frac{\partial \omega}{\partial p} - \beta v_g$$

$$\Rightarrow \quad \frac{\partial}{\partial t} \left(\frac{1}{f_0} \nabla^2 \Phi \right) + \mathbf{u}_g \cdot \nabla \left(\frac{1}{f_0} \nabla^2 \Phi \right) = f_0 \frac{\partial \omega}{\partial p} - \beta v_g$$

QG Geopotential Tendency

Definition: The **geopotential tendency** of a flow is the Eulerian rate of change in geopotential with respect to time.

$$\chi \equiv \frac{\partial \Phi}{\partial t}$$

$$\frac{1}{f_0}\nabla^2\chi = \underbrace{-\mathbf{u}_g \cdot \nabla\left(\frac{1}{f_0}\nabla^2\Phi + f\right)}_{\text{Geostrophic Terms}} + \underbrace{f_0\frac{\partial\omega}{\partial p}}_{\text{Ageostrophic Term}}$$

QG Geopotential Tendency

$$\frac{1}{f_0}\nabla^2\chi = -\mathbf{u}_g \cdot \nabla\left(\frac{1}{f_0}\nabla^2\Phi + f\right) + f_0\frac{\partial\omega}{\partial p}$$

Geostrophic Wind

$$\mathbf{u}_g = \frac{1}{f_0}\mathbf{k}\times\nabla\Phi$$

$$\frac{1}{f_0}\nabla^2\chi = -\left(\frac{1}{f_0}\mathbf{k}\times\nabla\Phi\right)\cdot\nabla\left(\frac{1}{f_0}\nabla^2\Phi + f\right) + f_0\frac{\partial\omega}{\partial p}$$

Only unknowns are Φ and ω

The previous equation for geopotential tendency used these three equations.

Geostrophic Wind

$$\mathbf{u}_g = \frac{1}{f_0}\mathbf{k} \times \nabla \Phi$$

Momentum Equation

$$\frac{D_g \mathbf{u}_g}{Dt} = -f_0 \mathbf{k} \times \mathbf{u}_a - \beta y \mathbf{k} \times \mathbf{u}_g$$

Continuity Equation

$$\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} + \frac{\partial \omega}{\partial p} = 0$$

Thermodynamic Equation

$$\left(\frac{\partial}{\partial t} + \mathbf{u}_g \cdot \nabla\right)\left(-\frac{\partial \Phi}{\partial p}\right) - \sigma\omega = \frac{\kappa J}{p}$$

For barotropic systems (ω=0) this leads to a closed prognostic equation for geopotential.

QG Geopotential Tendency

Geostrophic Wind

$$\mathbf{u}_g = \frac{1}{f_0}\mathbf{k} \times \nabla\Phi$$

Momentum Equation

$$\frac{D_g \mathbf{u}_g}{Dt} = -f_0 \mathbf{k} \times \mathbf{u}_a - \beta y \mathbf{k} \times \mathbf{u}_g$$

Continuity Equation

$$\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} + \frac{\partial \omega}{\partial p} = 0$$

Thermodynamic Equation

$$\left(\frac{\partial}{\partial t} + \mathbf{u}_g \cdot \nabla\right)\left(-\frac{\partial \Phi}{\partial p}\right) - \sigma\omega = \frac{\kappa J}{p}$$

However, for baroclinic systems (horizontal temperature / density gradients), all four equations must be used.

QG Geopotential Tendency

Replace this term

$$\frac{1}{f_0}\nabla^2\chi = -\left(\frac{1}{f_0}\mathbf{k}\times\nabla\Phi\right)\cdot\nabla\left(\frac{1}{f_0}\nabla^2\Phi + f\right) + f_0\frac{\partial\omega}{\partial p}$$

Starting from the QG thermodynamic equation:

$$-\frac{\partial}{\partial t}\left(\frac{\partial\Phi}{\partial p}\right) - \mathbf{u}_g\cdot\nabla\left(\frac{\partial\Phi}{\partial p}\right) - \sigma\omega = \frac{\kappa J}{p}$$

Flip derivatives

$$\underbrace{-\frac{\partial}{\partial p}\left(\frac{\partial\Phi}{\partial t}\right)}_{\frac{\partial\chi}{\partial p}} - \mathbf{u}_g\cdot\nabla\left(\frac{\partial\Phi}{\partial p}\right) - \sigma\omega - \frac{\kappa J}{p} = 0$$

QG Geopotential Tendency

$$-\frac{\partial \chi}{\partial p} - \mathbf{u}_g \cdot \nabla \left(\frac{\partial \Phi}{\partial p} \right) - \sigma\omega - \frac{\kappa J}{p} = 0$$

$$\frac{f_0}{\sigma}\frac{\partial}{\partial p} \Rightarrow \frac{f_0}{\sigma}\frac{\partial}{\partial p}\left[-\frac{\partial \chi}{\partial p} - \mathbf{u}_g \cdot \nabla \left(\frac{\partial \Phi}{\partial p} \right) - \sigma\omega - \frac{\kappa J}{p} \right] = 0$$

Assume stability parameter σ is constant with respect to p

$$\frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial \chi}{\partial p} \right) = -\frac{\partial}{\partial p}\left[\frac{f_0}{\sigma}\mathbf{u}_g \cdot \nabla \left(\frac{\partial \Phi}{\partial p} \right) \right] - f_0 \frac{\partial \omega}{\partial p} - f_0 \frac{\partial}{\partial p}\left(\frac{\kappa J}{\sigma p} \right)$$

QG Geopotential Tendency

Geopotential tendency equation from before:

$$\frac{1}{f_0}\nabla^2\chi = -\left(\frac{1}{f_0}\mathbf{k}\times\nabla\Phi\right)\cdot\nabla\left(\frac{1}{f_0}\nabla^2\Phi + f\right)\boxed{+ f_0\frac{\partial\omega}{\partial p}}$$

From QG thermodynamic equation:

$$\frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial\chi}{\partial p}\right) = -\frac{\partial}{\partial p}\left[\frac{f_0}{\sigma}\mathbf{u}_g\cdot\nabla\left(\frac{\partial\Phi}{\partial p}\right)\right]\boxed{- f_0\frac{\partial\omega}{\partial p}} - f_0\frac{\partial}{\partial p}\left(\frac{\kappa J}{\sigma p}\right)$$

Different sign!

Adding these two equations will eliminate the omega term.

QG Geopotential Tendency

QG Geopotential Tendency Eq'n (Adiabatic)

$$\underbrace{\left[\nabla^2 + \frac{\partial}{\partial p}\left(\frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\right)\right]\chi}_{\text{Local geopotential tendency}} = \underbrace{-f_0 \mathbf{u}_g \cdot \nabla\left(\frac{1}{f_0}\nabla^2\Phi + f\right)}_{\text{Proportional to absolute vorticity advection}} - \frac{\partial}{\partial p}\underbrace{\left[-\frac{f_0^2}{\sigma}\mathbf{u}_g \cdot \nabla\left(-\frac{\partial \Phi}{\partial p}\right)\right]}_{\text{Proportional to (differential) thickness (temperature) advection}}$$

Only the initial distribution of Φ needs to be known!

QG Geopotential Tendency

For wave-like flows, second derivatives of χ are proportional to $-\chi$.

For example, $\chi = \sin x \Rightarrow \frac{\partial^2 \chi}{\partial x^2} = -\sin x = -\chi$

$$\Rightarrow \left[\nabla^2 + \frac{\partial}{\partial p}\left(\frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\right)\right]\chi \approx c \cdot (-\chi)$$

QG Equations

Question: What's the point?

We want to describe the evolution of two key features of the atmosphere:

- **Large-scale waves** (in particular, the connection between large-scale waves and geopotential)
- **Midlatitude cyclones** (that is, the development of low pressure systems in the lower troposphere)

QG Geopotential Tendency

Let's look at the **absolute vorticity** advection term

$$\left[\nabla^2 + \frac{\partial}{\partial p}\left(\frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\right)\right]\chi = -f_0 \mathbf{u}_g \cdot \nabla \left(\frac{1}{f_0}\nabla^2\Phi + f\right)$$

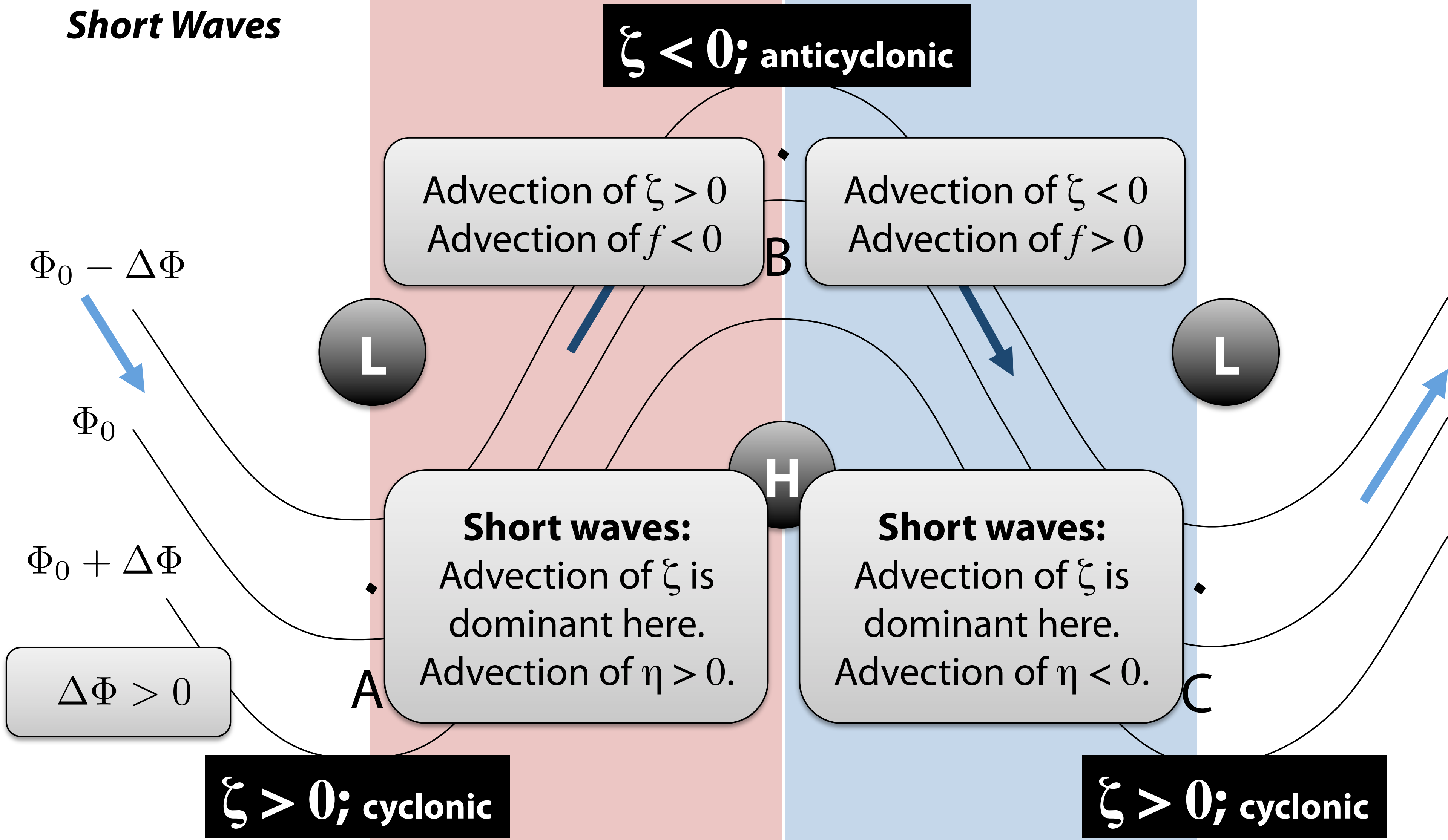
Relative vorticity

Planetary vorticity

$$-\chi = -\frac{\partial \Phi}{\partial t} \approx \underbrace{-f_0 \mathbf{u}_g \cdot \nabla \left(\frac{1}{f_0}\nabla^2\Phi + f\right)}_{\text{Absolute vorticity advection (times } f_0\text{)}}$$

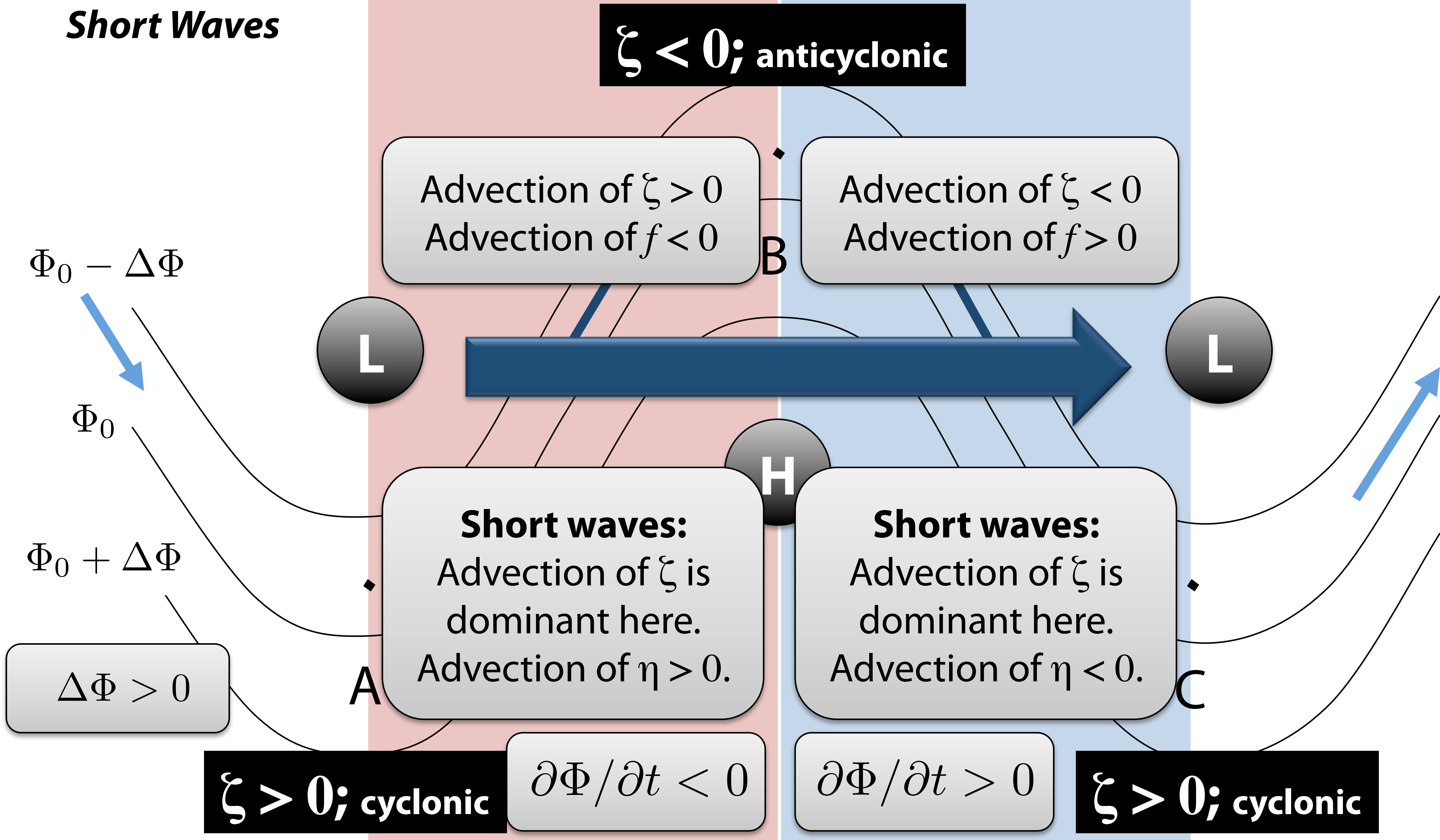
An Upper Tropospheric Wave

Short Waves



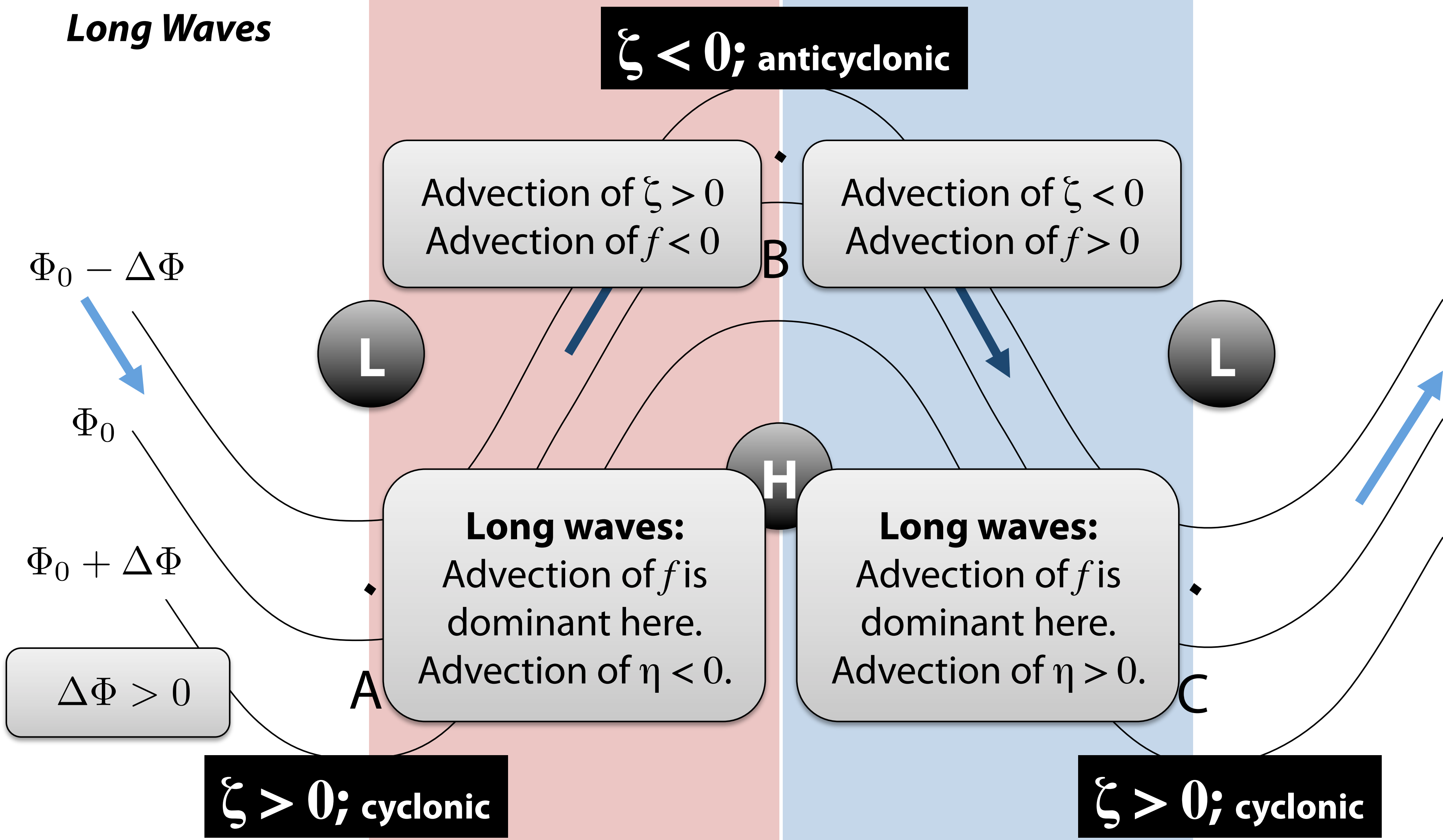
An Upper Tropospheric Wave

Short Waves



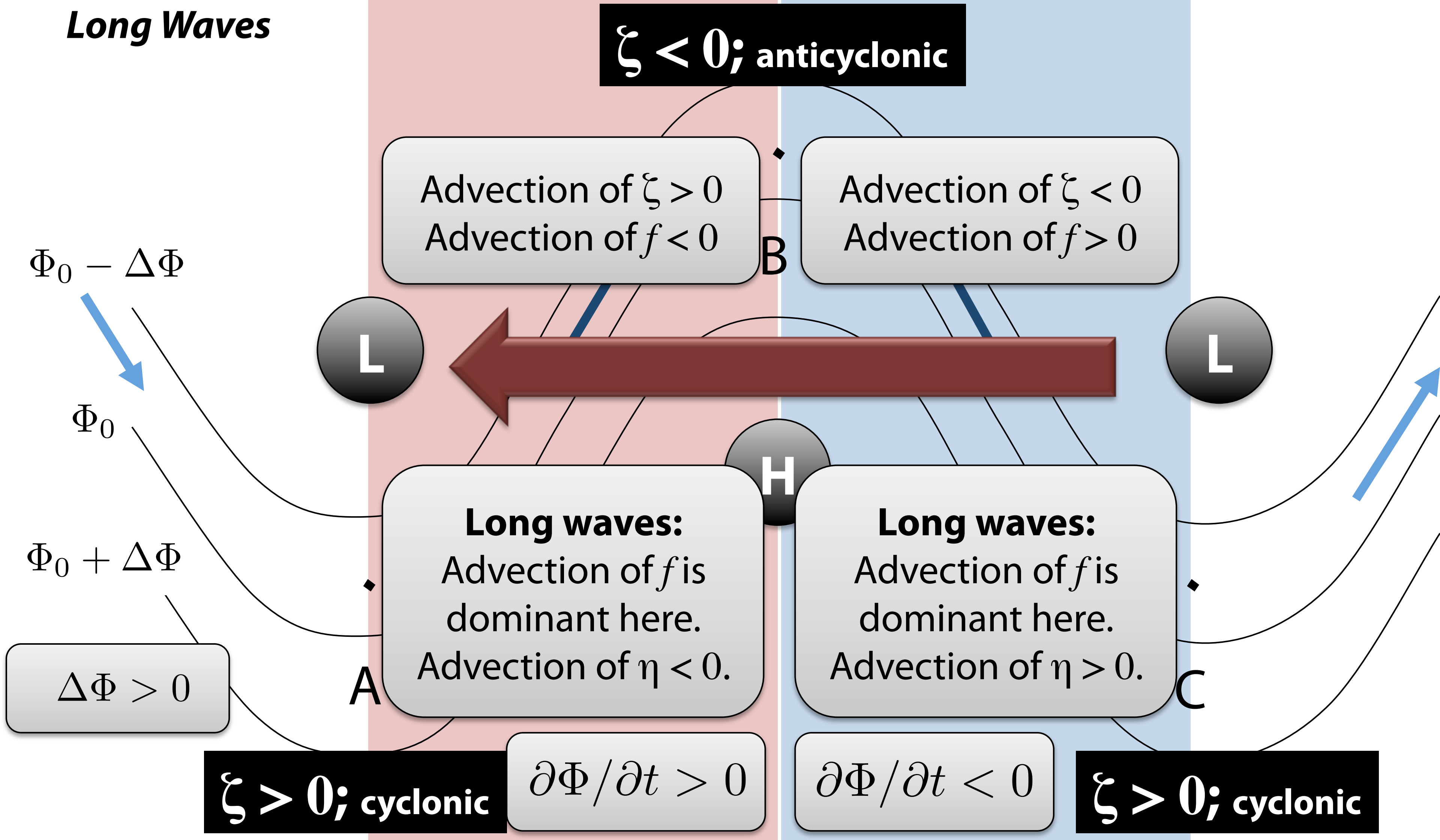
An Upper Tropospheric Wave

Long Waves

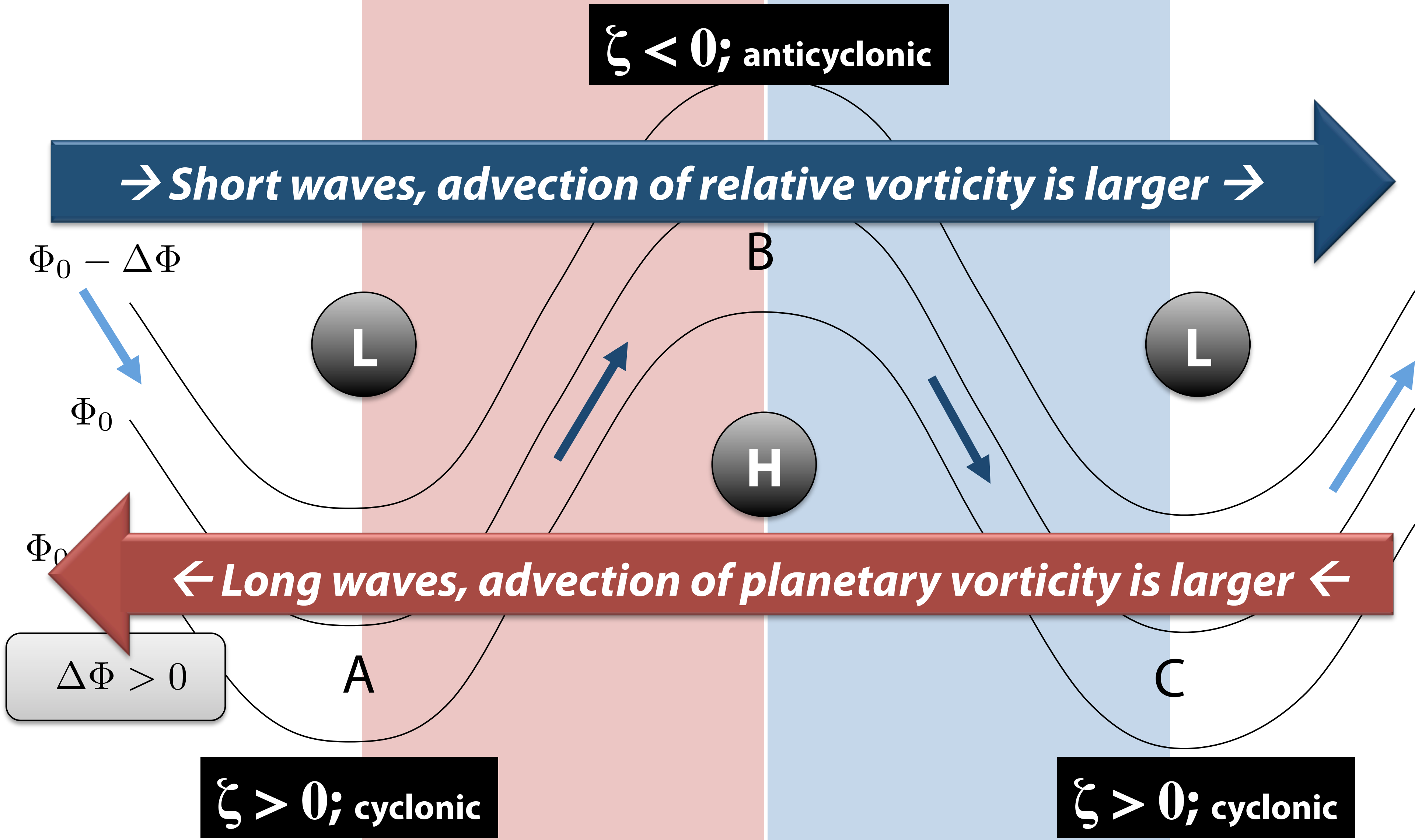


An Upper Tropospheric Wave

Long Waves



An Upper Tropospheric Wave



QG Equations

Question: What's the point?

We want to describe the evolution of two key features of the atmosphere:

- **Large-scale waves** (in particular, the connection between large-scale waves and geopotential)
- **Midlatitude cyclones** (that is, the development of low pressure systems in the lower troposphere)

QG Geopotential Tendency

Let's look at the **differential temperature** advection:

Differential

$$\chi = \frac{\partial \Phi}{\partial t} \approx \frac{\partial}{\partial p} \underbrace{\left[-\mathbf{u}_g \cdot \nabla \left(-\frac{\partial \Phi}{\partial p} \right) \right]}_{\text{Thickness advection}} \approx \frac{\partial}{\partial p} \underbrace{(-\mathbf{u}_g \cdot \nabla T)}_{\text{Temperature advection}}$$

Question: Why is thickness advection proportional to temperature advection?

QG Geopotential Tendency

Differential temperature advection:

$$\chi = \frac{\partial \Phi}{\partial t} \approx \frac{\partial}{\partial p}\left[-\mathbf{u}_g \cdot \nabla \left(-\frac{\partial \Phi}{\partial p}\right)\right] \approx \frac{\partial}{\partial p}(-\mathbf{u}_g \cdot \nabla T)$$

$$\Rightarrow \quad \chi \approx -\frac{\partial}{\partial z}\left(-\mathbf{u}_g \cdot \nabla T\right)$$

Differential temperature advection below 500 hPa:

+ (builds a ridge, increasing geopotential) in case of **decreasing warm air advection** with height

– (deepens a trough, decreasing geopotential) in case of **decreasing cold air advection** with height

Warm Fronts

- … are broader in shape than cold fronts
- … tend to move more slowly than cold fronts
- … have precipitation spread out over a larger distance

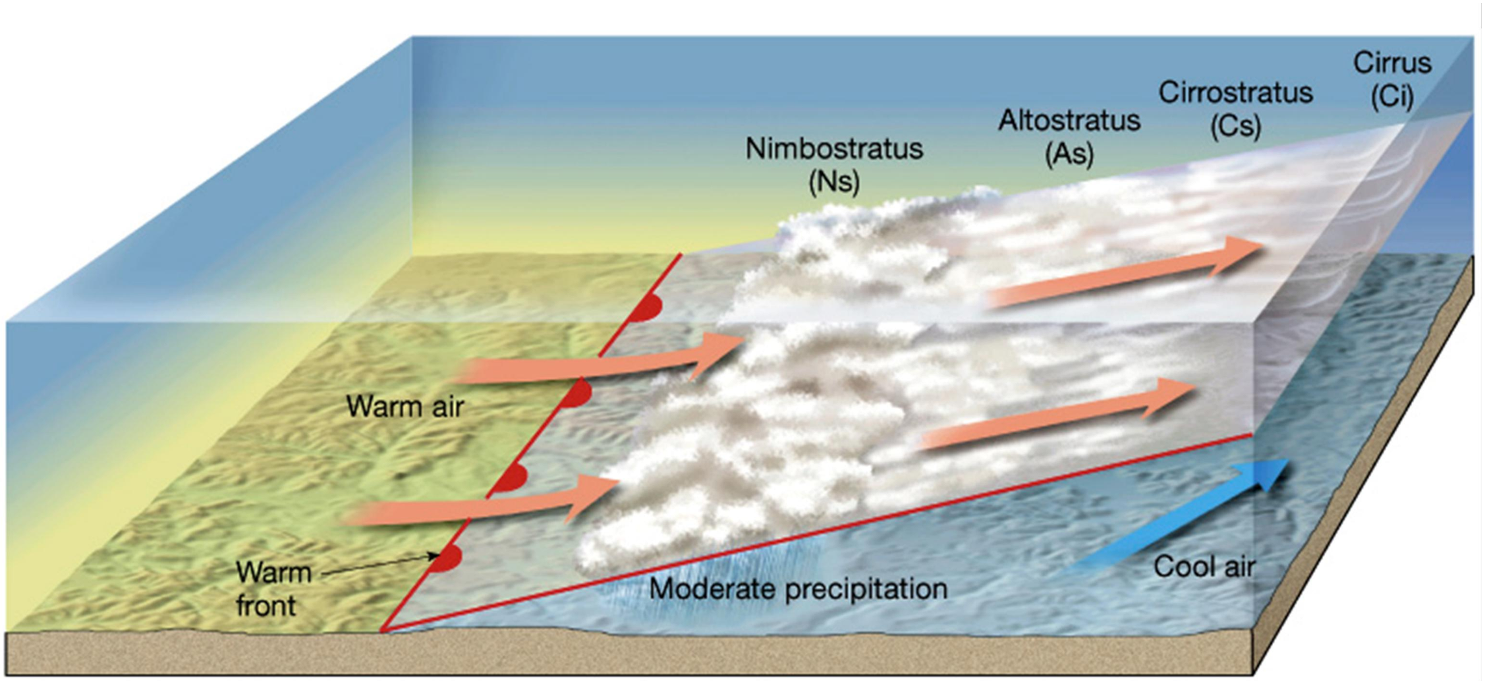


Figure 9.6 in *The Atmosphere, 8th edition*, Lutgens and Tarbuck, 8th edition, 2001.

Cold Fronts

- … are vertically steep
- … tend to travel faster than warm fronts
- … are associated with strong storms at boundary

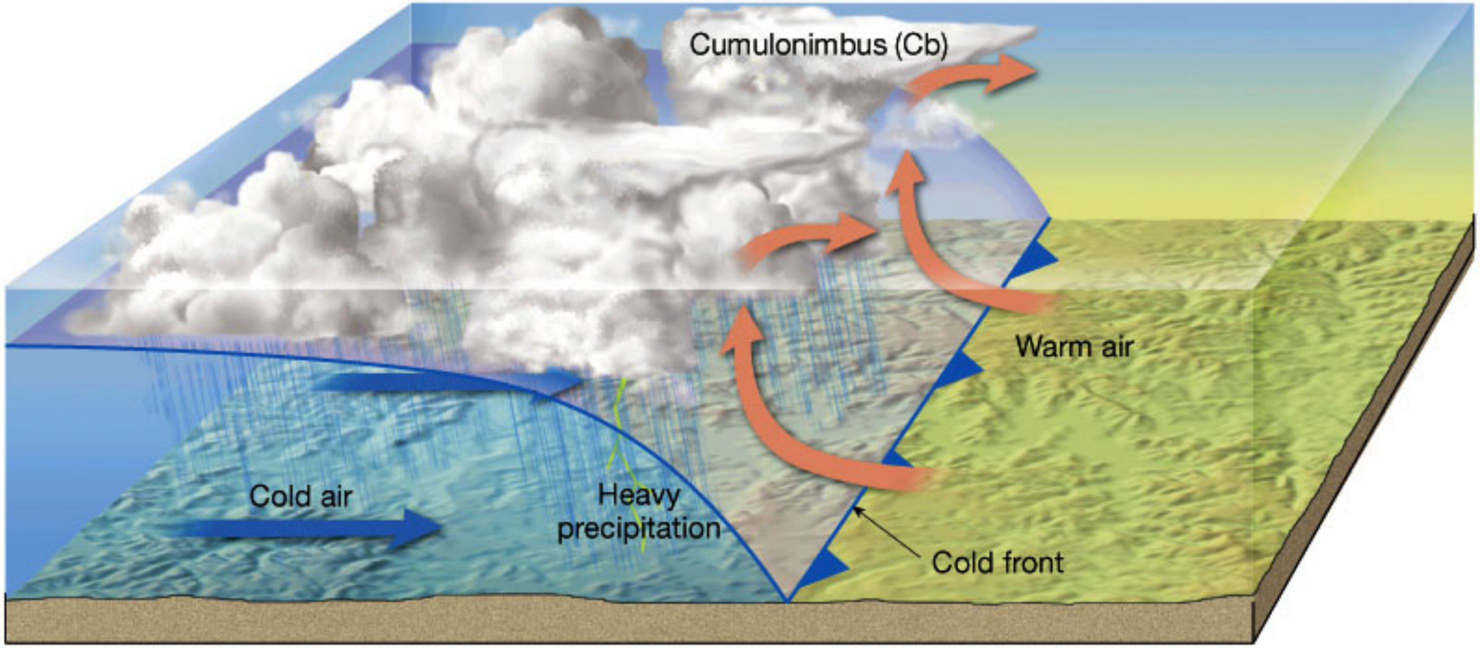


Figure 9.6 in *The Atmosphere, 8th edition*, Lutgens and Tarbuck, 8th edition, 2001.

Extratropical Cyclones

Extratropical Cylones are important for driving weather in the mid-latitudes. They are closely related to weather fronts.

Particularly strong extratropical systems are responsible for large-scale storm systems.

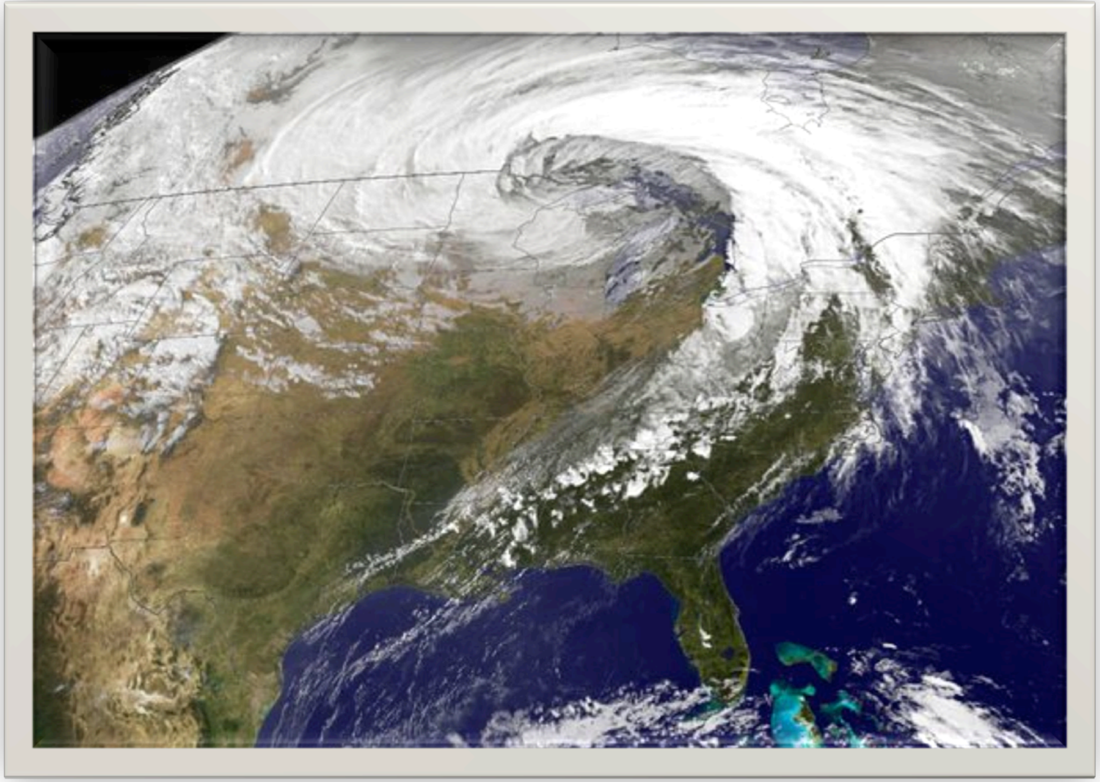
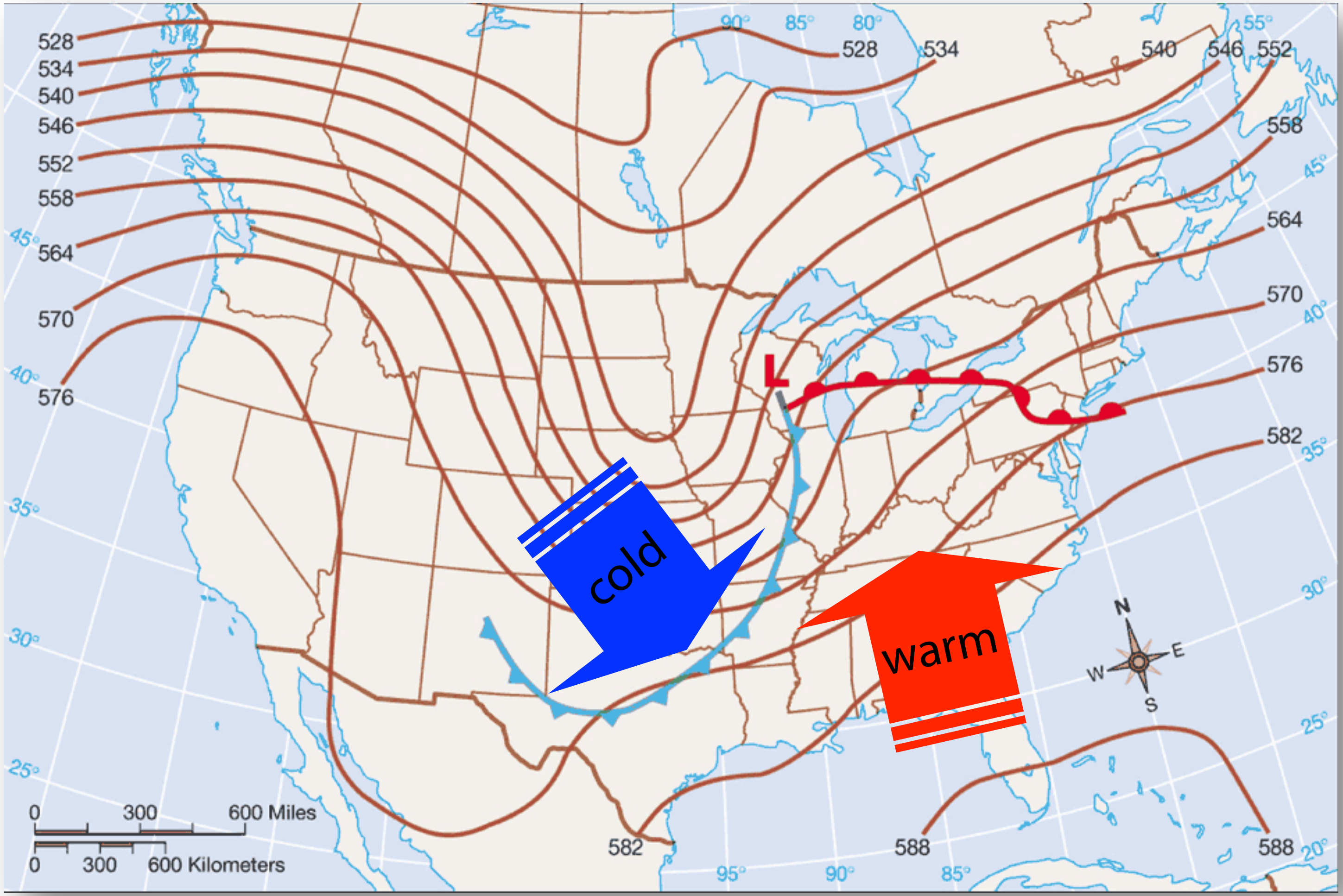


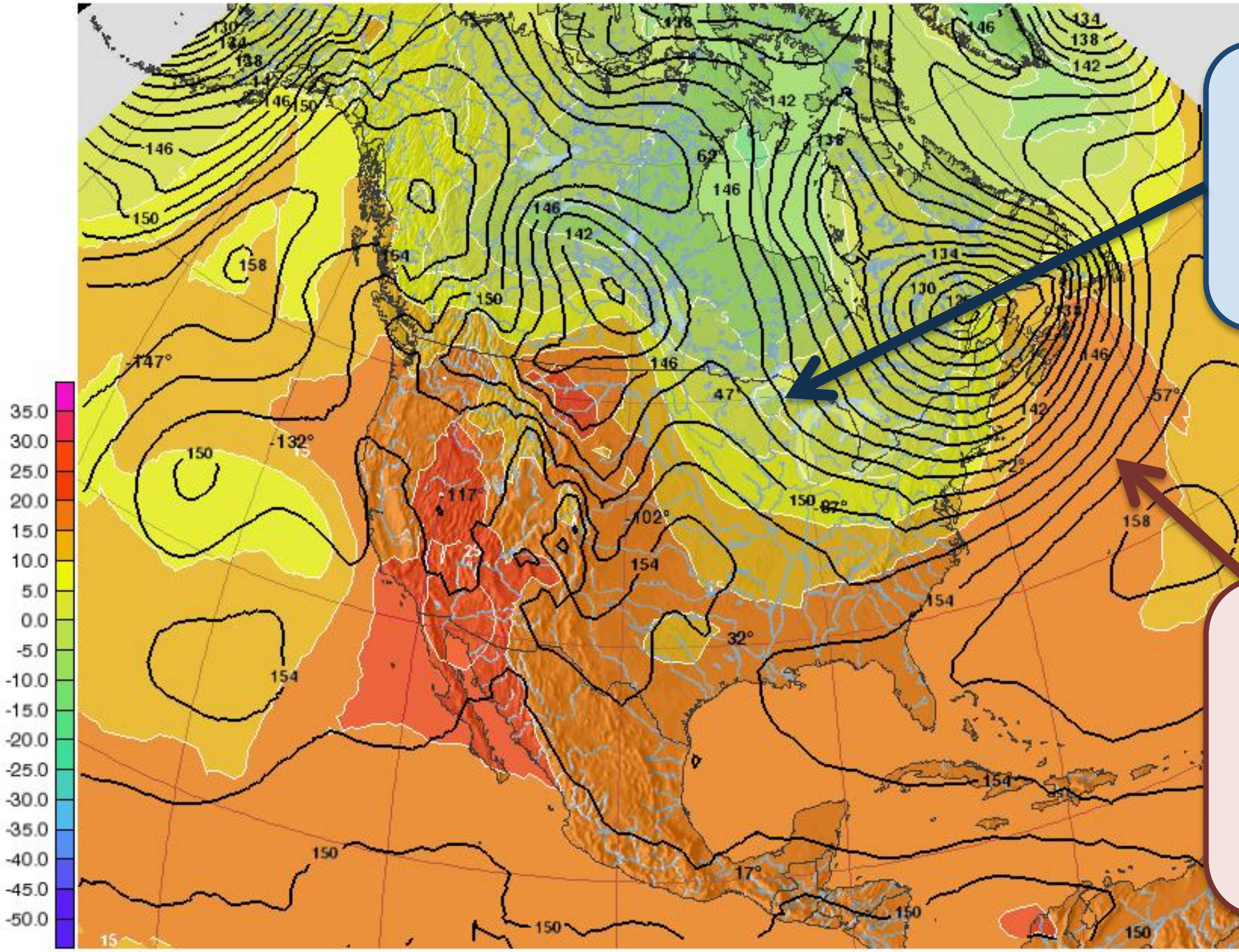
Figure: Extratropical Cyclones are associated with severe winter storm systems, and are particularly relevant for the US Northwest and Northern Europe.

Atmospheric Wave Motion



Real Baroclinic Disturbances

850 hPa Temperature and Geopotential Thickness



(c) 2007 by meteoblue TM & NOAA/NCEP - www.meteoblue.com

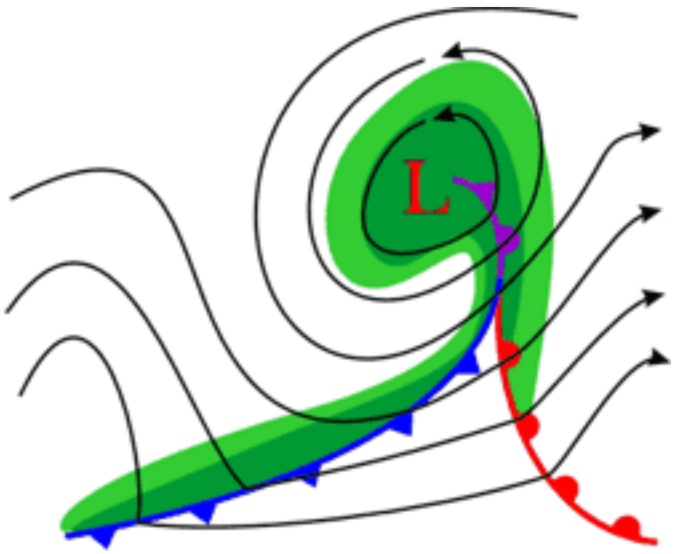
Cold Air Advection west of the surface low enhances the trough

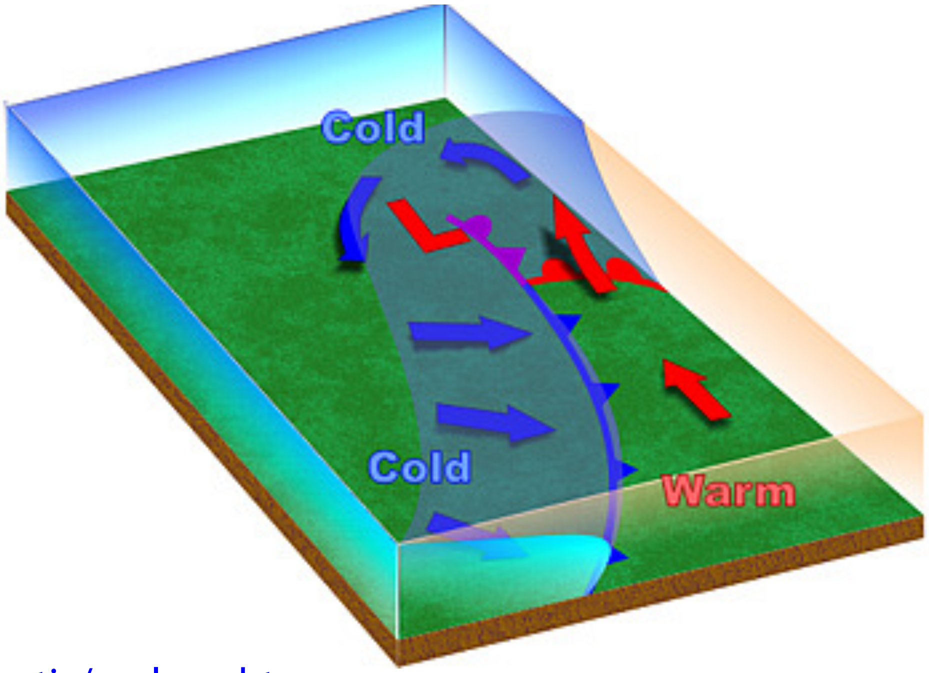
Warm Air Advection east of the surface low enhances the ridge

Norwegian Cyclone Model

The wave becomes a mature low pressure system, while the cold front, moving faster than the warm front, "catches up" with the warm front. As the cold front overtakes the warm front, an occluded front forms.

Overhead view





http://www.srh.weather.gov/jetstream/synoptic/cyclone.htm

QG Geopotential Tendency

Thickness advection $\propto -\frac{\partial}{\partial z}\left[-\mathbf{u}_g \cdot \nabla T\right]$

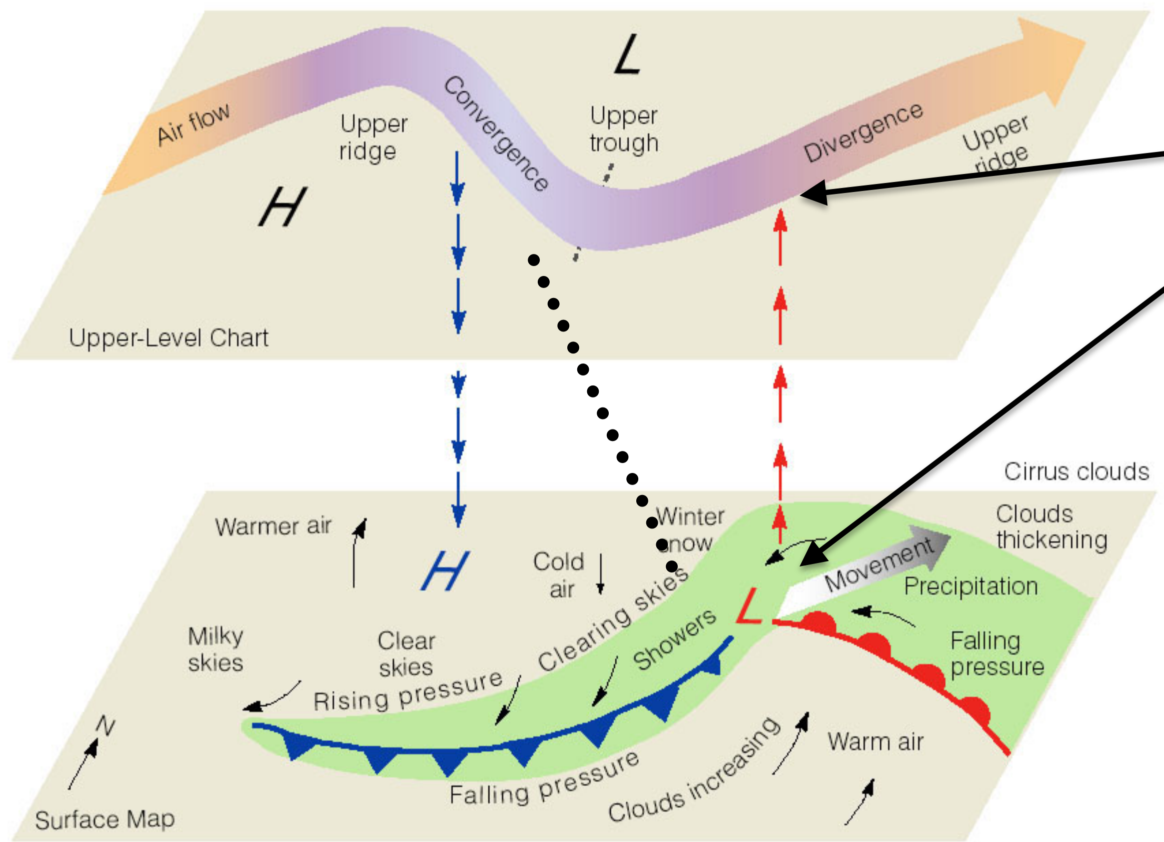
Vorticity advection $\propto -\mathbf{u}_g \cdot \nabla\left(\zeta_g + f\right)$

We see that:

- Geostrophic advection of geostrophic vorticity causes waves to ***propagate***
- The vertical difference in temperature (thickness) advection causes waves to ***amplify***

Baroclinic Instability

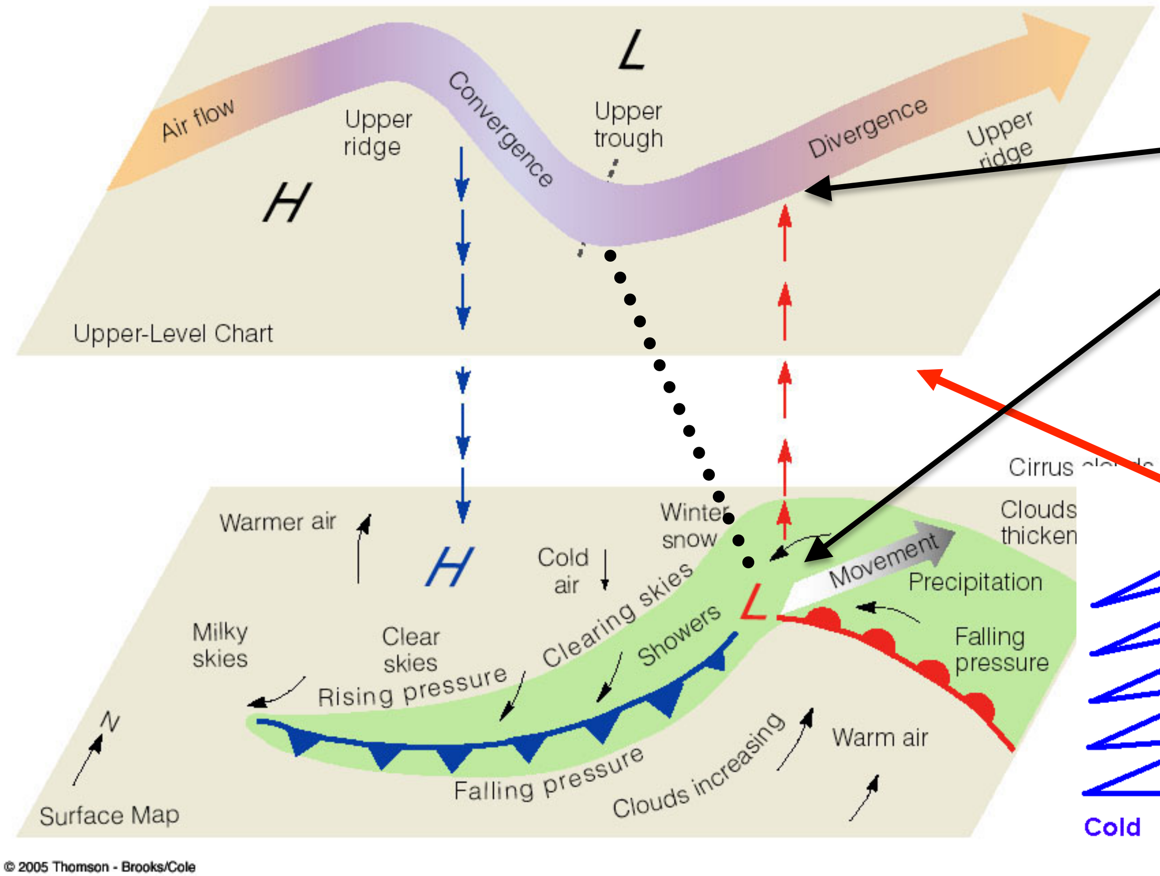
Upper troposphere and surface



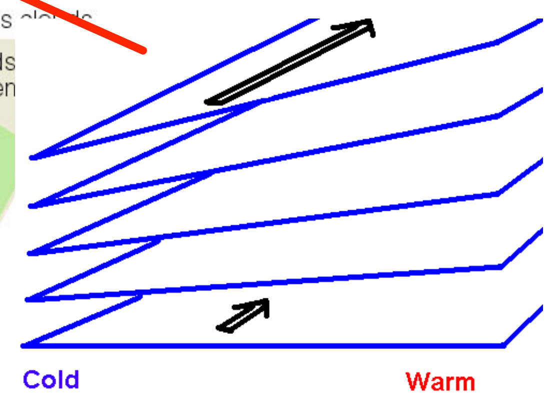
Divergence over low enhances surface low and increases vorticity

Baroclinic Instability

Upper troposphere and surface

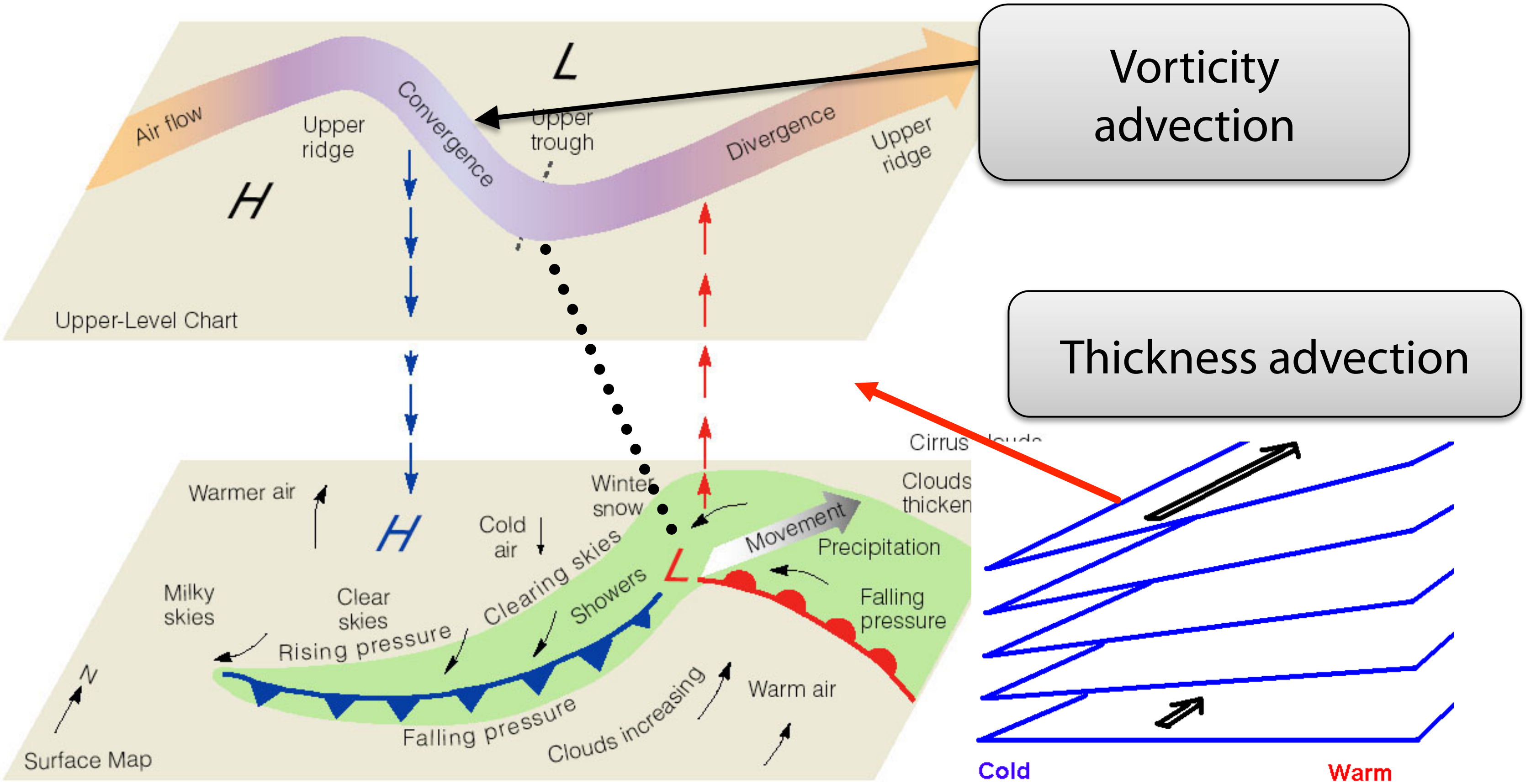


Vertical stretching increases vorticity



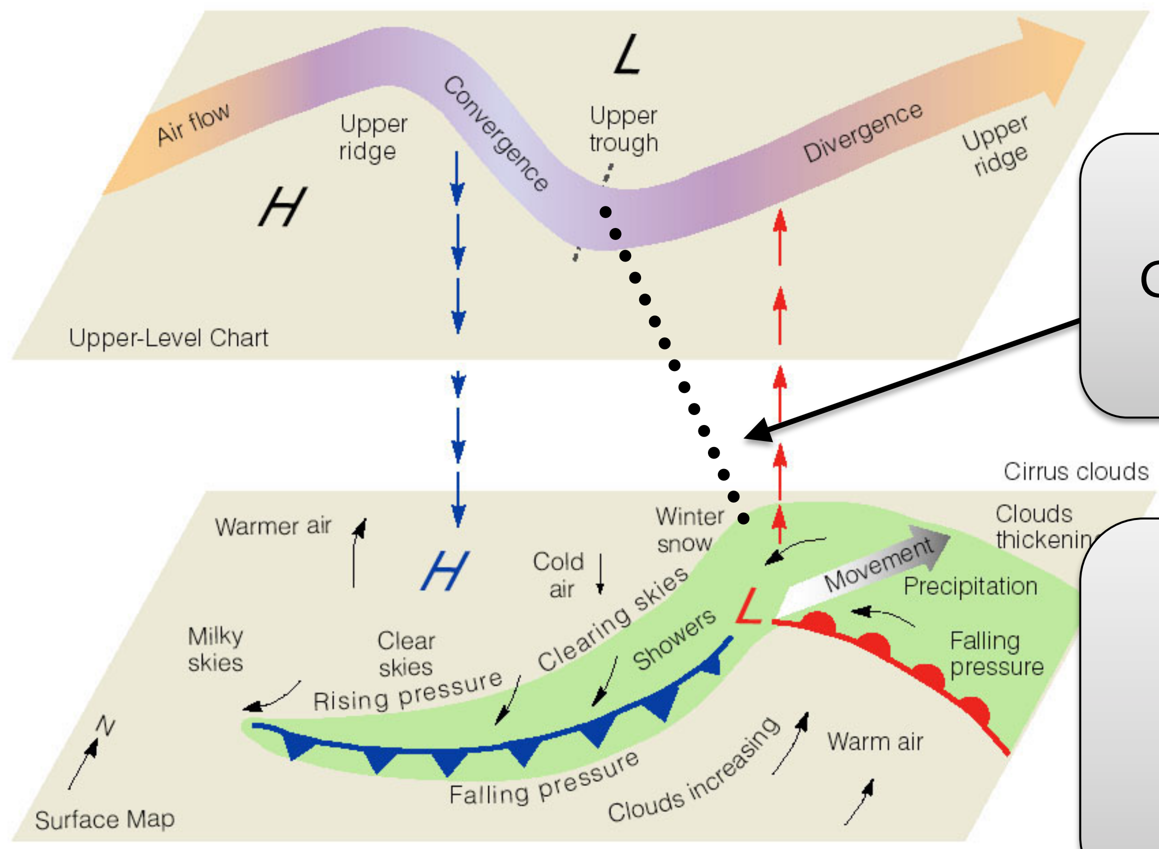
Baroclinic Instability

Upper troposphere and surface



Baroclinic Instability

Upper troposphere and surface



Note tilt with height. Conservation of energy at work here.

It turns out that a disturbance with a westward tilt leads to poleward transfer of energy