# Quasi-Geostrophic Theory

## Chapter 5

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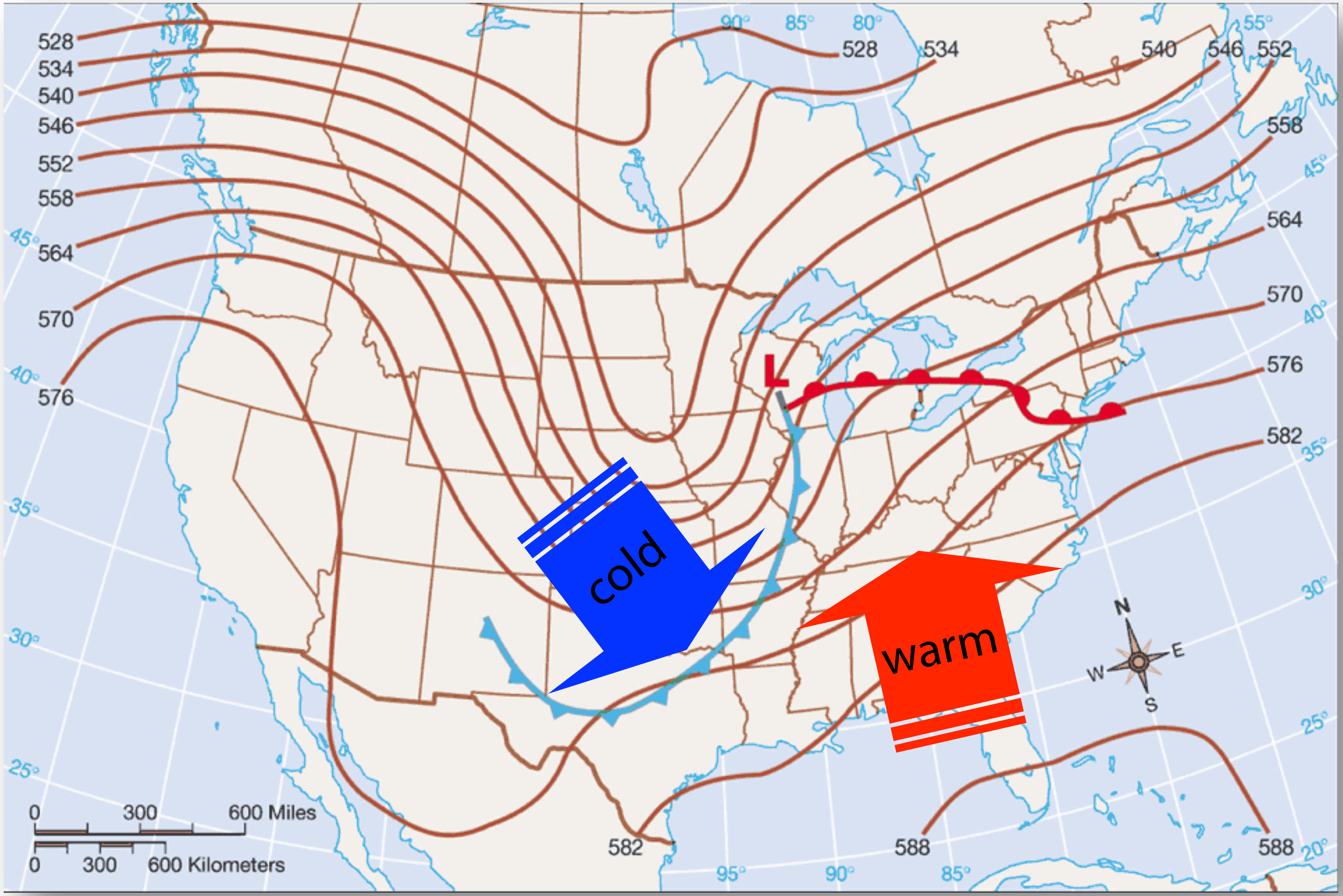
# Part 3: Synoptic Scale Waves

# Atmospheric Wave Motion



# QG Equations

**Question:** What's the point?

We want to describe the evolution of two key features of the atmosphere:

- **Large-scale waves** (in particular, the connection between large-scale waves and geopotential)
- **Midlatitude cyclones** (that is, the development of low pressure systems in the lower troposphere)

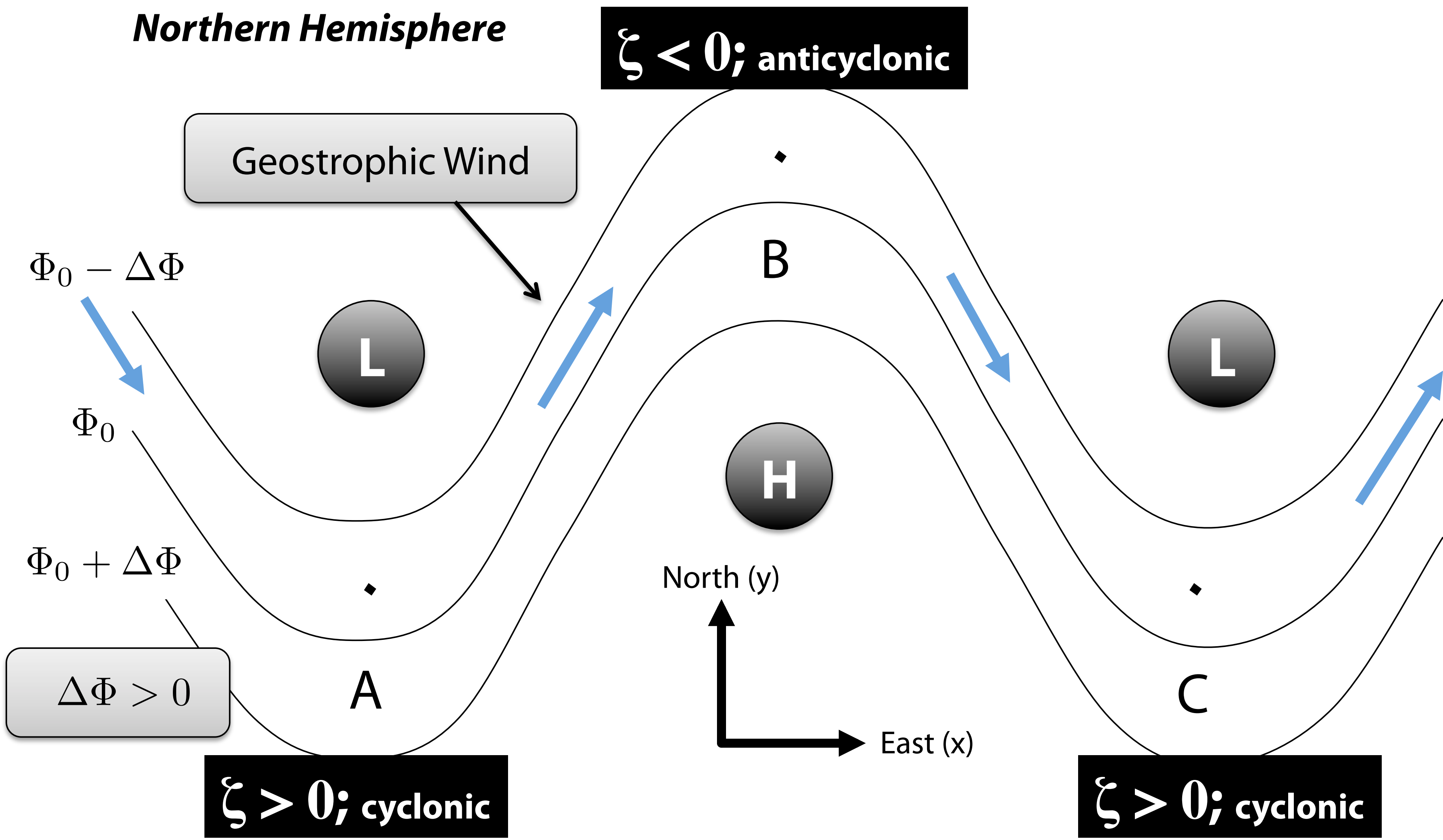
## QG Vorticity Equation

$$\frac{\partial \zeta_g}{\partial t} = f_0 \frac{\partial \omega}{\partial p} - \mathbf{u}_g \cdot \nabla(\zeta_g + f)$$

Consider only the last term

$$\underbrace{-\mathbf{u}_g \cdot \nabla(\zeta_g + f)}_{\text{Advection of absolute vorticity}} = \underbrace{-\mathbf{u}_g \cdot \nabla \zeta_g}_{\text{Advection of relative vorticity}} \underbrace{- \beta v_g}_{\text{Advection of planetary vorticity}}$$

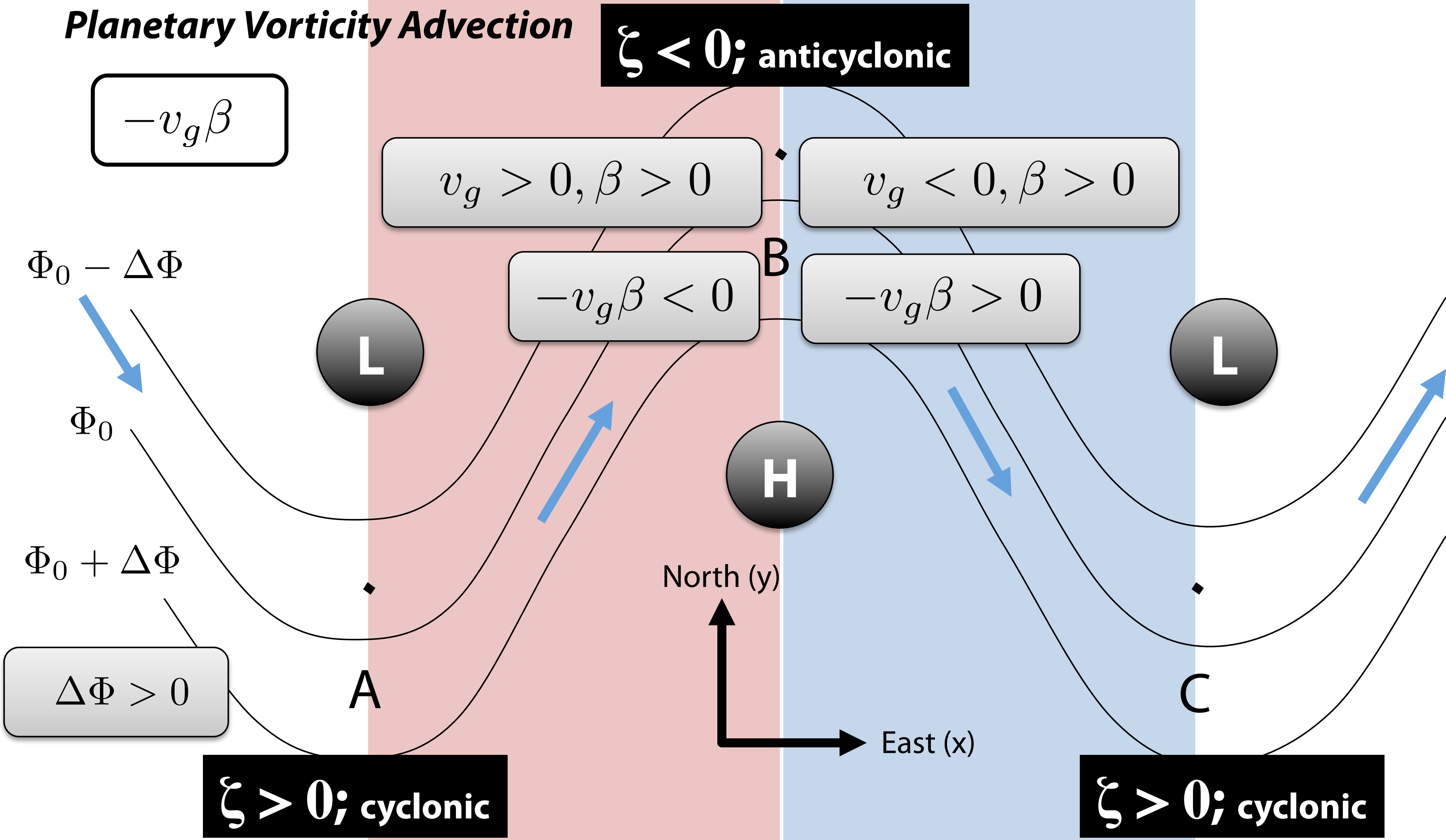
# An Upper Tropospheric Wave



# An Upper Tropospheric Wave

***Planetary Vorticity Advection***

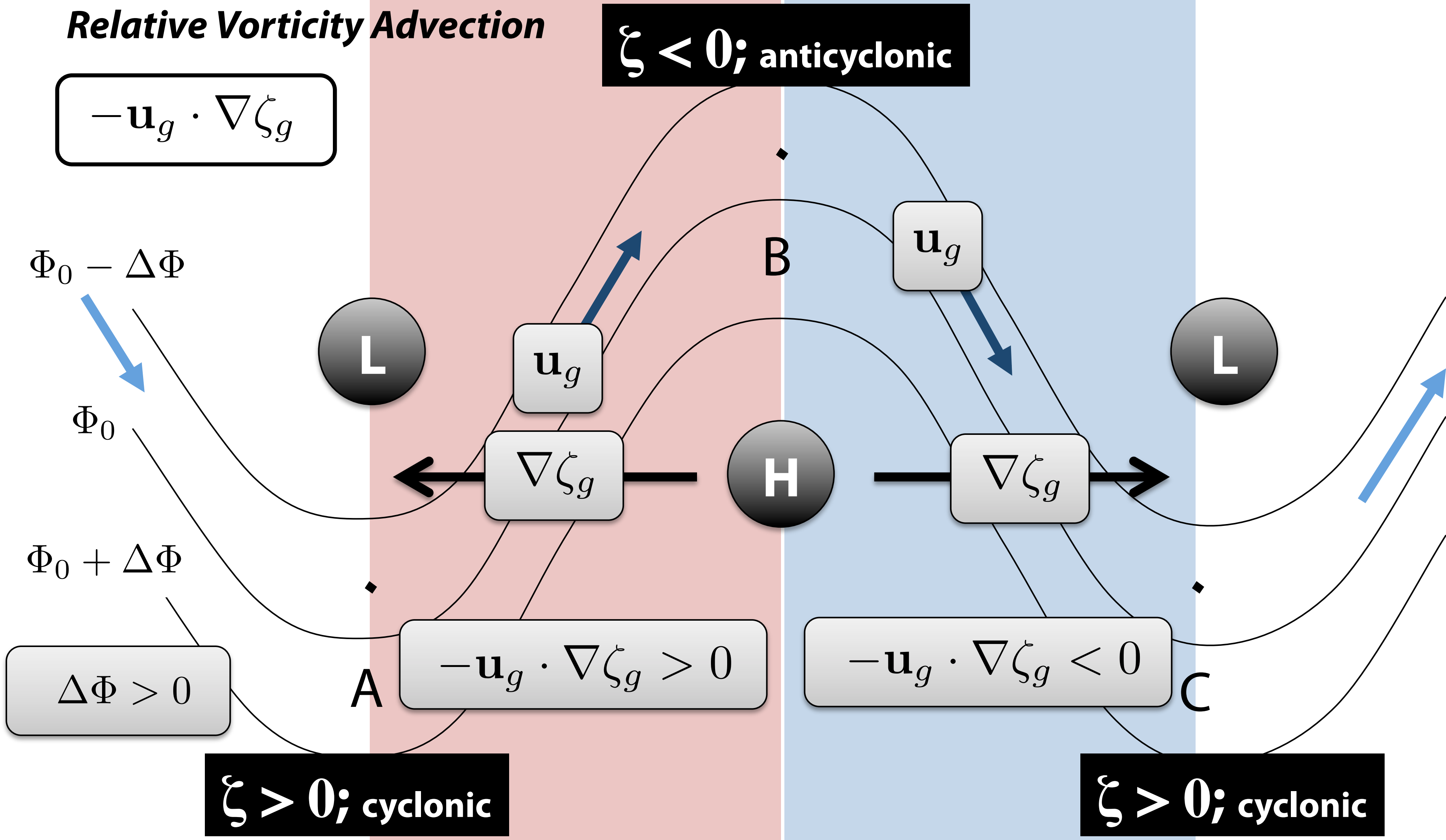
$$\boxed{-v_g\beta}$$



# An Upper Tropospheric Wave

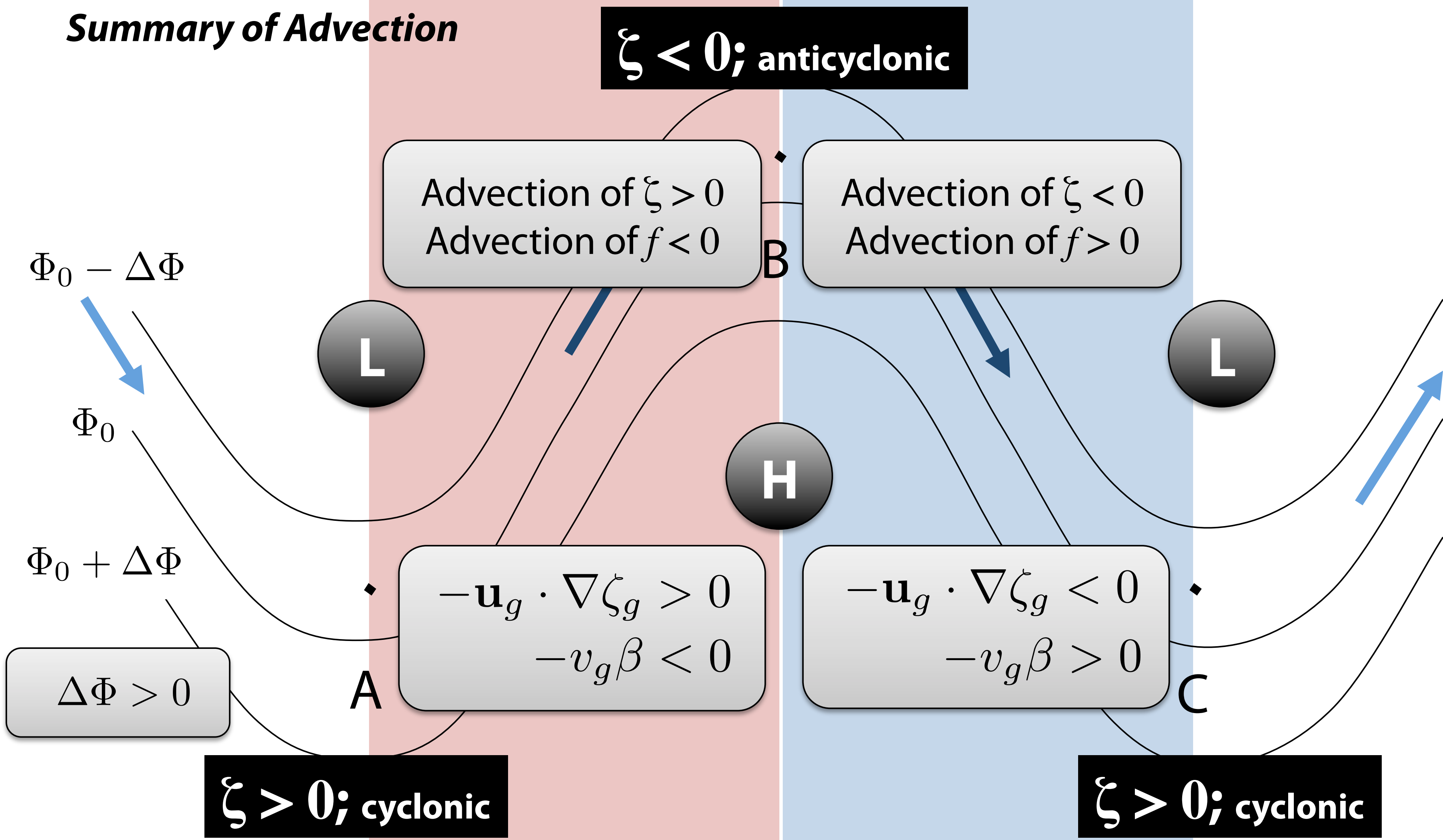
***Relative Vorticity Advection***

$$\boxed{-\mathbf{u}_g \cdot \nabla \zeta_g}$$



# An Upper Tropospheric Wave

### *Summary of Advection*



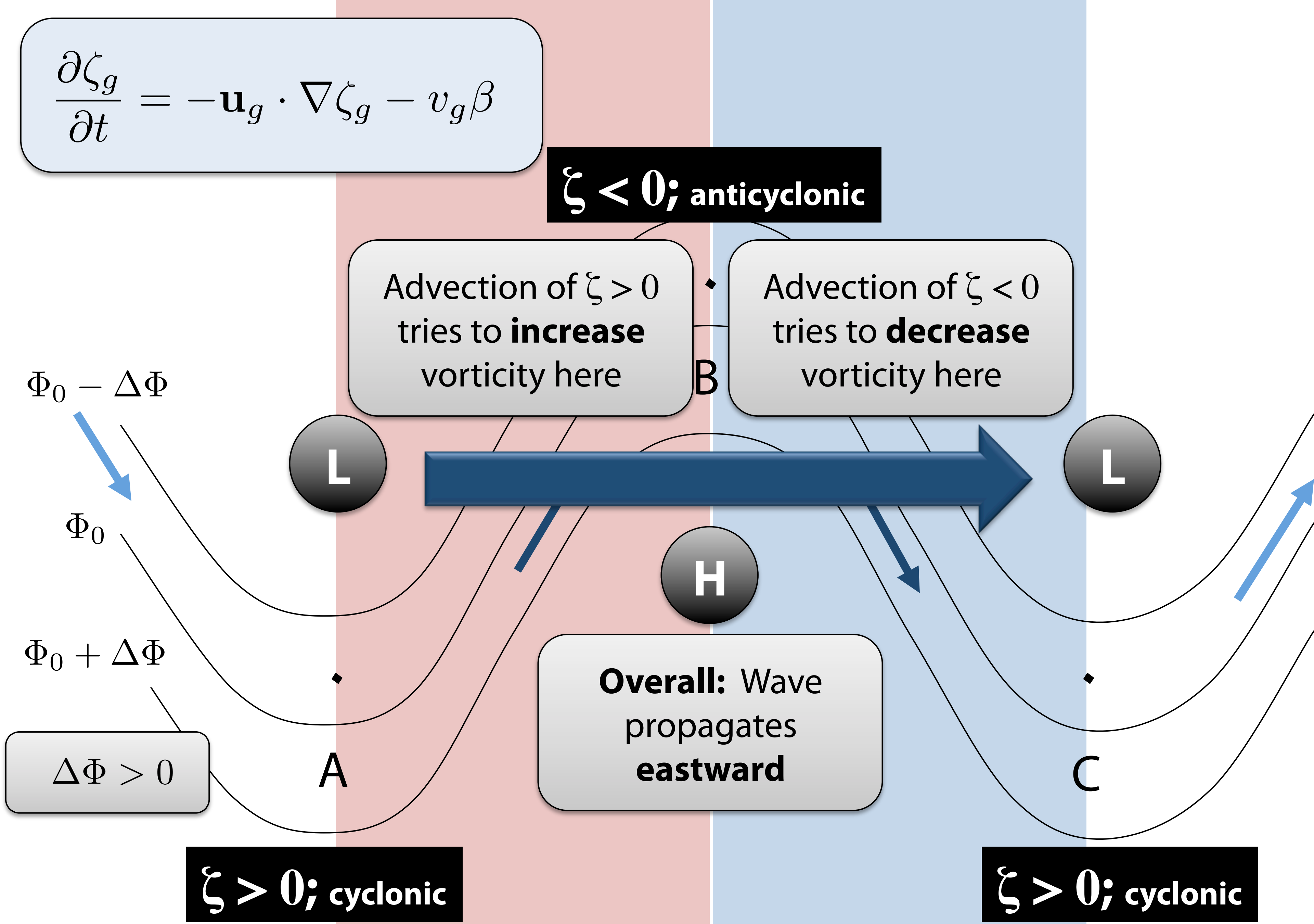
# An Upper Tropospheric Wave

## Summary of Advection

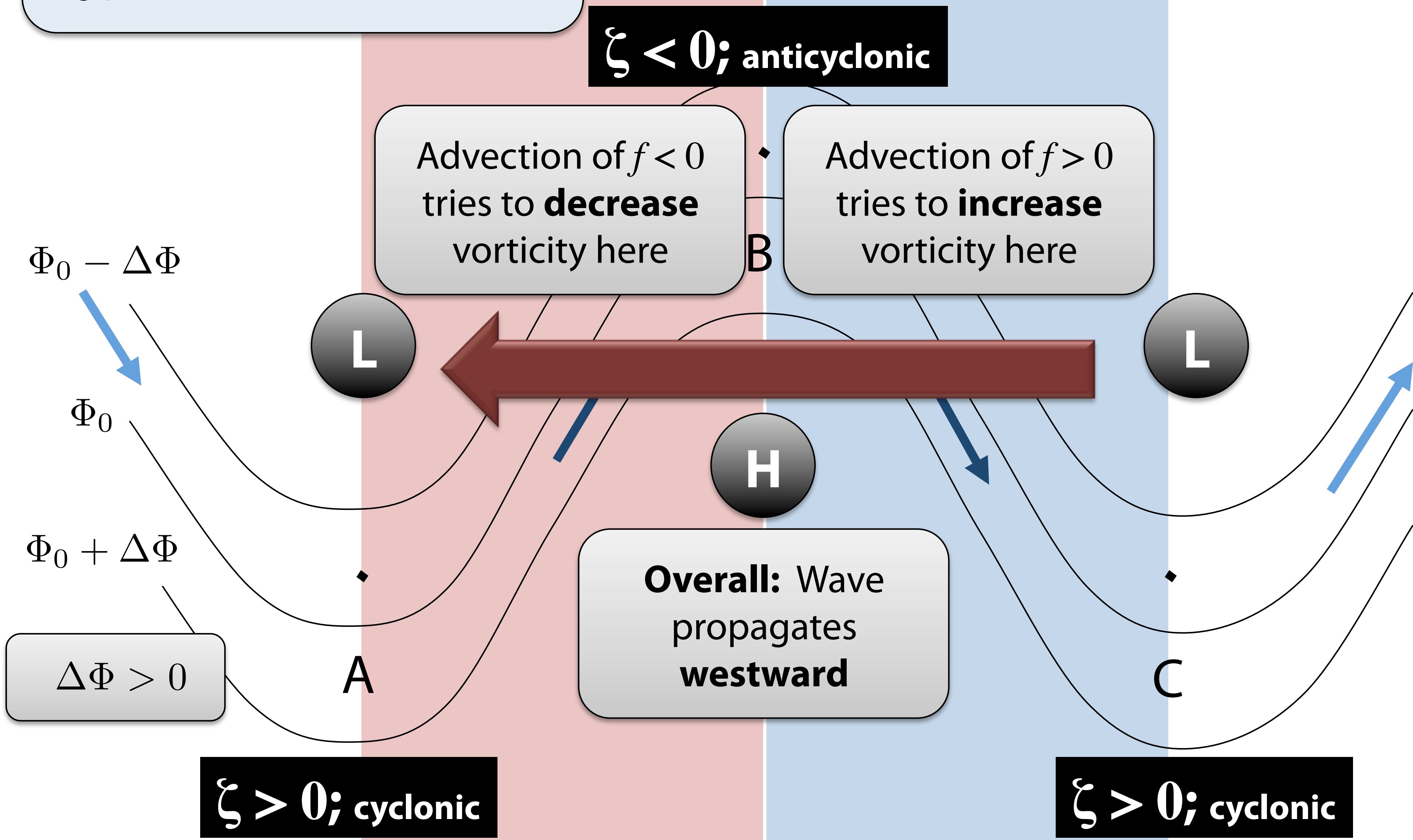
Planetary and relative vorticity advection in a wave **oppose each other**.

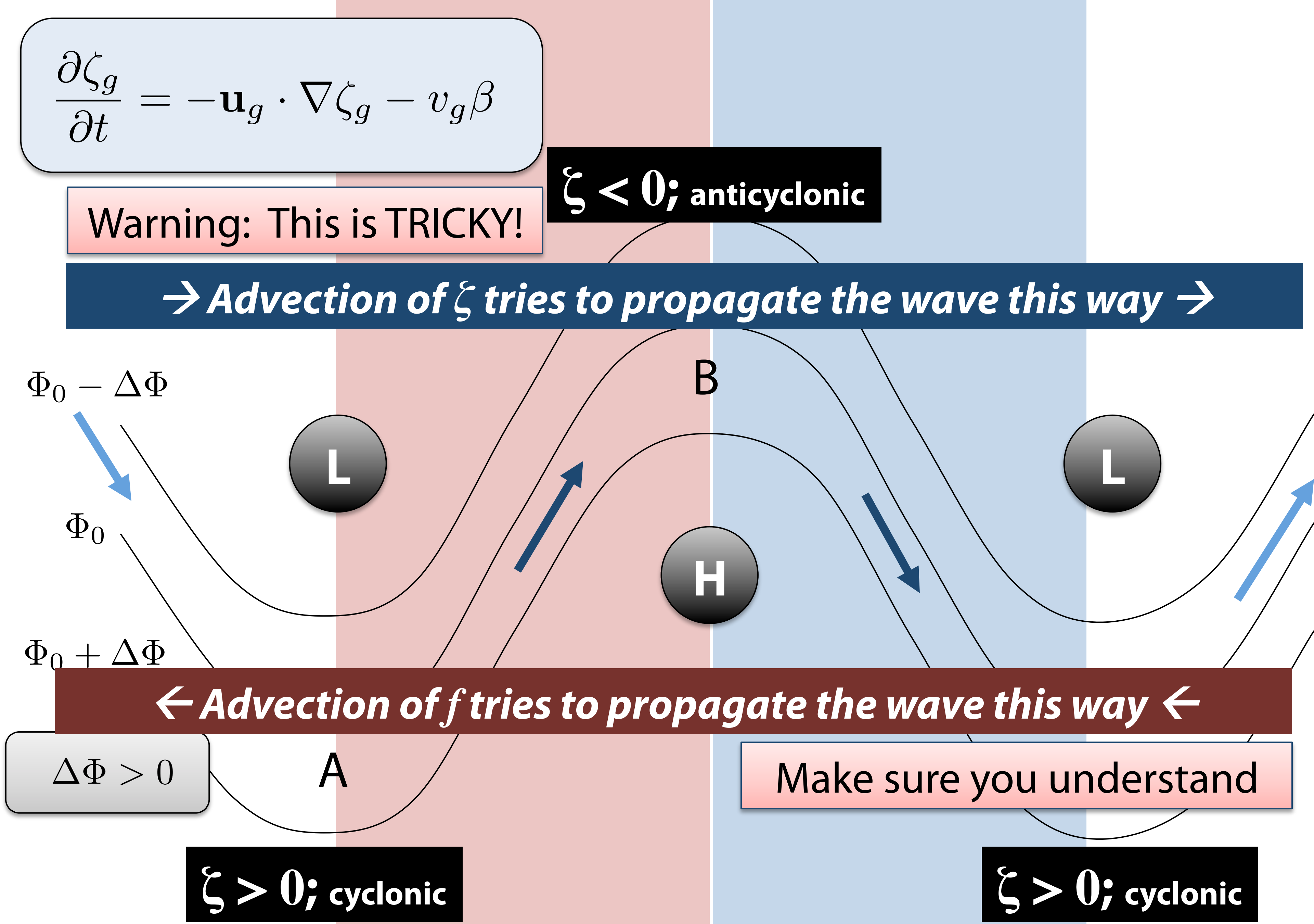
This is consistent with the balance that we intuitively derived from the conservation of absolute vorticity over the mountain.

**Question:** What does this do to the wave?


$$\frac{\partial \zeta_g}{\partial t} = -\mathbf{u}_g \cdot \nabla \zeta_g - v_g \beta$$
$\zeta < 0$; anticyclonic
Advection of $\zeta > 0$ tries to **increase** vorticity here
Advection of $\zeta < 0$ tries to **decrease** vorticity here
$\Phi_0 - \Delta\Phi$
$\Phi_0$
$\Phi_0 + \Delta\Phi$
$\Delta\Phi > 0$
B
L
L
H
A
C
**Overall:** Wave propagates **eastward**
$\zeta > 0$; cyclonic
$\zeta > 0$; cyclonic

$$\frac{\partial \zeta_g}{\partial t} = -\mathbf{u}_g \cdot \nabla \zeta_g - v_g \beta$$




$\frac{\partial \zeta_g}{\partial t} = -\mathbf{u}_g \cdot \nabla \zeta_g - v_g \beta$
Warning: This is TRICKY!
$\zeta < 0$; anticyclonic
→ Advection of $\zeta$ tries to propagate the wave this way →
$\Phi_0 - \Delta\Phi$
B
L
L
$\Phi_0$
H
$\Phi_0 + \Delta\Phi$
← Advection of $f$ tries to propagate the wave this way ←
$\Delta\Phi > 0$
A
Make sure you understand
$\zeta > 0$; cyclonic
$\zeta > 0$; cyclonic

# *Vorticity & Geopotential*

**QG Vorticity Equation**

$$\frac{\partial \zeta_g}{\partial t} = f_0 \frac{\partial \omega}{\partial p} - \mathbf{u}_g \cdot \nabla \zeta_g - v_g \beta$$

Advection Terms

**Geostrophic Wind**

$$u_g = -\frac{1}{f_0}\frac{\partial \Phi}{\partial y} \qquad v_g = \frac{1}{f_0}\frac{\partial \Phi}{\partial x}$$

**Geostrophic Vorticity**

Relates geostrophic vorticity and geopotential

$$\zeta_g = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = \frac{1}{f_0}\left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}\right) = \frac{1}{f_0}\nabla^2 \Phi$$

# An Upper Tropospheric Wave

***Assume geopotential is a wave***

$$\Phi(x,y) = \underbrace{\Phi_0(p)}_{\text{Background geopotential as a function of pressure}} - \underbrace{f_0 \overline{u}(p) y}_{\text{Contribution of zonal mean wind to geostrophic balance}} + \underbrace{f_0 A(p) \sin(kx) \cos(\ell y)}_{\text{Wave-like component}}$$

With wavenumbers $k$ and $\ell$ and wavelengths $L_x$ and $L_y$:

$$k = \frac{2\pi}{L_x} \qquad \ell = \frac{2\pi}{L_y}$$

# An Upper Tropospheric Wave

***Assume geopotential is a wave***

$$\Phi(x,y) = \Phi_0(p) - f_0\overline{u}(p)y + f_0 A(p)\sin(kx)\cos(\ell y)$$

The **geostrophic wind components** are then given by:

$$u_g = -\frac{1}{f_0}\frac{\partial\Phi}{\partial y} = \overline{u} + u_g' = \overline{u} + \ell A\sin(kx)\sin(\ell y)$$
$$v_g = \frac{1}{f_0}\frac{\partial\Phi}{\partial x} = v_g' = kA\cos(kx)\cos(\ell y)$$

# An Upper Tropospheric Wave

***Assume geopotential is a wave***

$$\Phi(x,y) = \Phi_0(p) - f_0\overline{u}(p)y + f_0 A(p)\sin(kx)\cos(\ell y)$$

$$u_g = -\frac{1}{f_0}\frac{\partial \Phi}{\partial y} = \overline{u} + u_g' = \overline{u} + \ell A \sin(kx)\sin(\ell y)$$

$$v_g = \frac{1}{f_0}\frac{\partial \Phi}{\partial x} = v_g' = kA\cos(kx)\cos(\ell y)$$

$$\zeta_g = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = -(k^2+\ell^2)A\sin(kx)\cos(\ell y)$$

# An Upper Tropospheric Wave

***Assume geopotential is a wave***

**Vorticity**

$$\zeta_g = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = -(k^2 + \ell^2) A \sin(kx) \cos(\ell y)$$

**Relative vorticity advection:**

$$-u_g \frac{\partial \zeta_g}{\partial x} - v_g \frac{\partial \zeta_g}{\partial y} \approx -\overline{u} \frac{\partial \zeta_g}{\partial x} = k\overline{u}(k^2 + \ell^2) A \cos(kx) \cos(\ell y)$$

**Planetary vorticity advection:**

$$-\beta v_g = -\beta k A \cos(kx) \cos(\ell y)$$

# An Upper Tropospheric Wave

***Assume geopotential is a wave***

Comparison of the planetary and relative advection terms tells us that the **relative strength of these terms depends on wavelength**.

$$\frac{\text{Planetary Vorticity Advection}}{\text{Relative Vorticity Advection}} = \frac{-\beta v_g}{-\mathbf{u}_g \cdot \nabla \zeta_g} = \frac{\beta}{\overline{u}\left[\left(\frac{2\pi}{L_x}\right)^2 + \left(\frac{2\pi}{L_y}\right)^2\right]}$$

# An Upper Tropospheric Wave

***Assume geopotential is a wave***

$$\frac{\text{Planetary Vorticity Advection}}{\text{Relative Vorticity Advection}} = \frac{-\beta v_g}{-\mathbf{u}_g \cdot \nabla \zeta_g} = \frac{\beta}{\overline{u}\left[\left(\frac{2\pi}{L_x}\right)^2 + \left(\frac{2\pi}{L_y}\right)^2\right]}$$

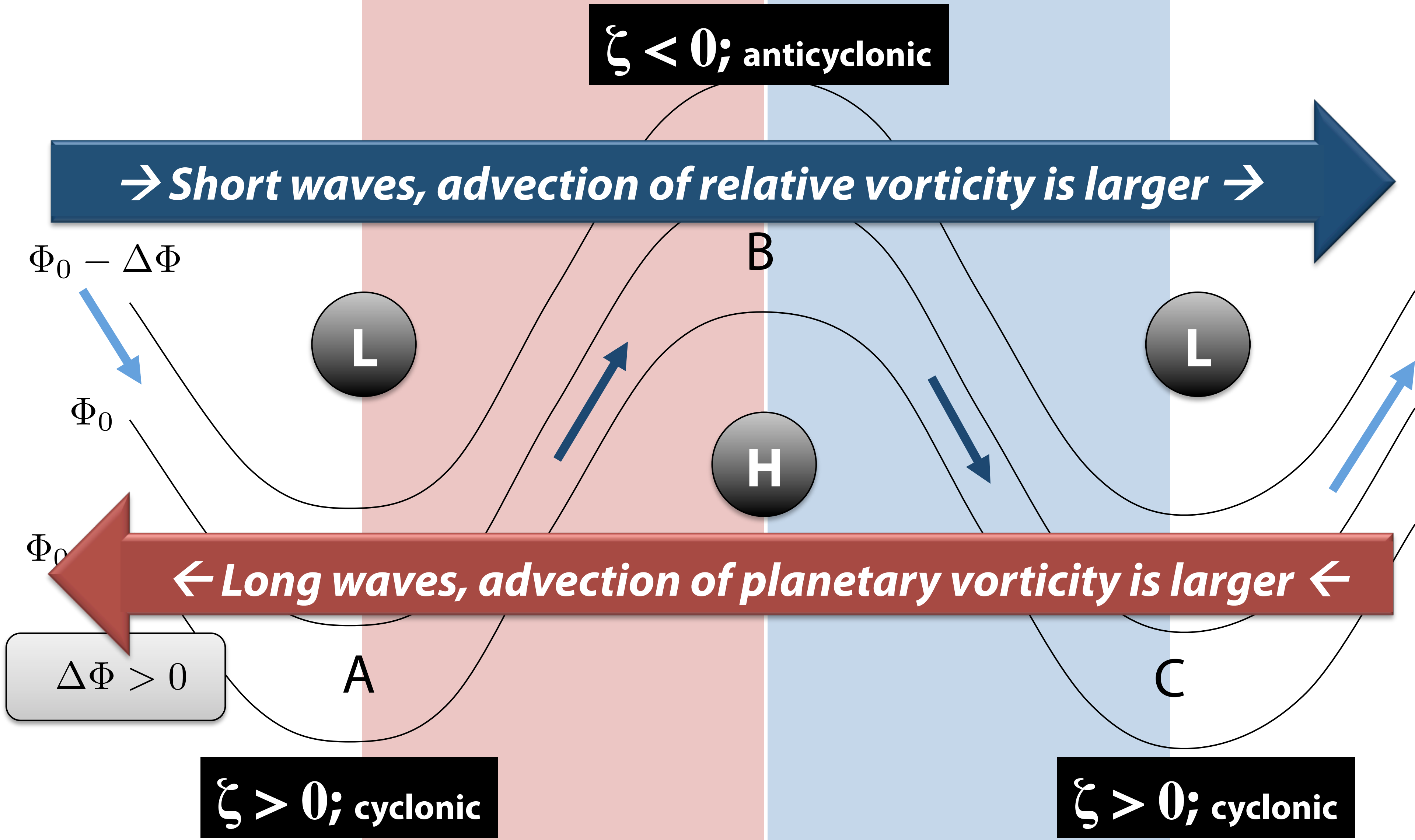
**Short waves:** $L_x$ and $L_y$ small: **Denominator is large**

***→ Short waves, advection of relative vorticity is larger →***

**Long waves:** $L_x$ and $L_y$ large: **Denominator is small**

***← Long waves, advection of planetary vorticity is larger ←***

# An Upper Tropospheric Wave



# *An Upper Tropospheric Wave*

### *Summary*

**Short waves**

- Strong curvature: **large** values of relative vorticity.
- Relatively **small** amplitude: Relatively **small** changes in coriolis parameter from trough to ridge.
- Advection of relative vorticity > advection of planetary vorticity
- **Wave propagates to the east**

**Long waves**

- Weak curvature: **small** values of relative vorticity.
- Relatively **large** amplitude: Relatively **large** changes in Coriolis parameter from trough to ridge.
- Advection of relative vorticity < advection of planetary vorticity
- **Wave propagates to the west** (depending on the mean wind speed)