Quasi-Geostrophic Theory

Chapter 5

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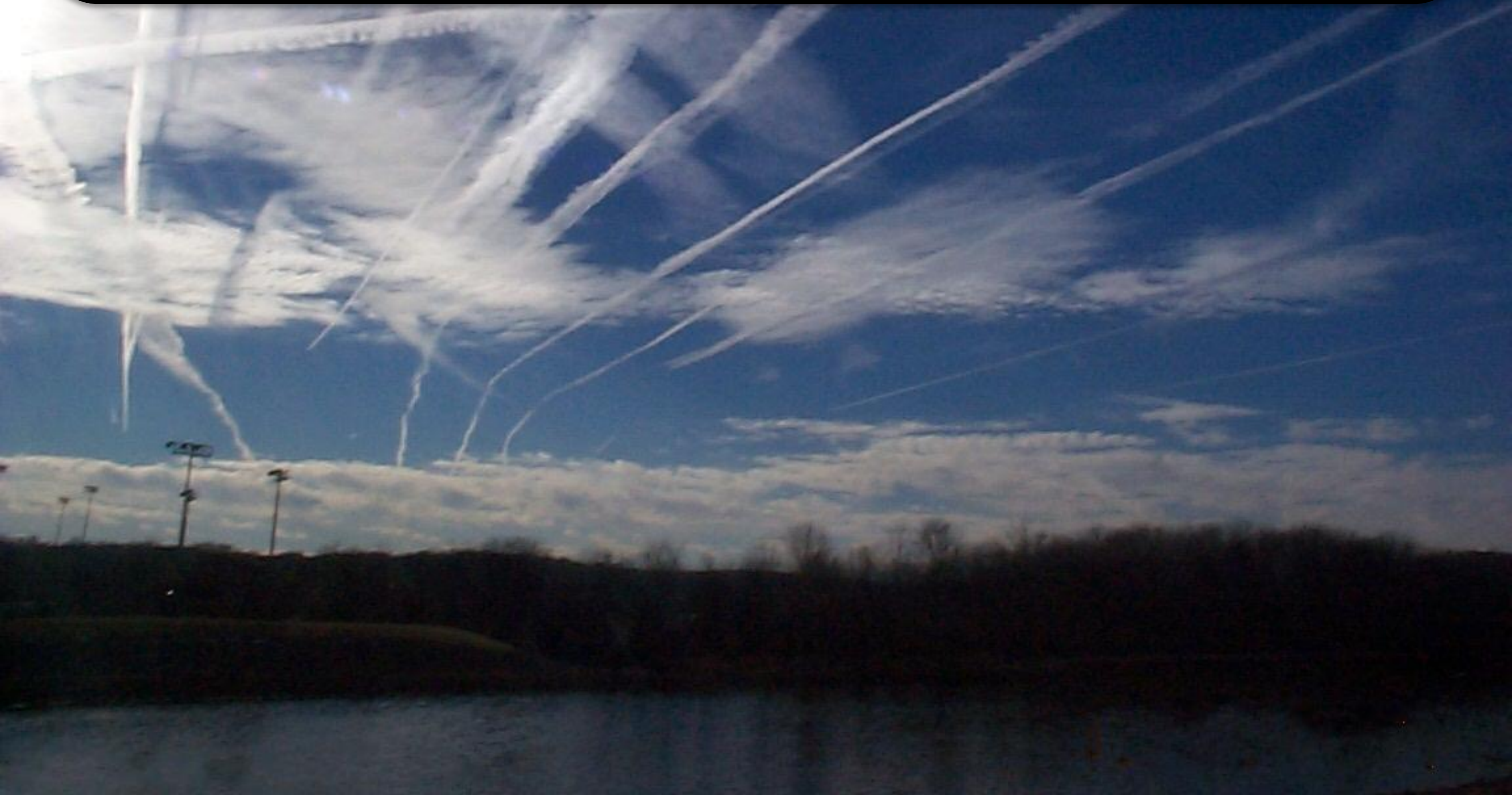
Part 2: The Equations of Quasi-Geostrophic Theory



Quasi-Geostrophic Theory

Observe: For **extratropical synoptic-scale** motions:

- Horizontal velocities approximately geostrophic (**quasi-geostrophic**)
- Atmosphere is approximately **hydrostatic**

This approach allows us to simplify the 3D equations of motion while retaining the time derivatives (prognostic equations)

Suitable model to forecast the dynamics of weather systems in midlatitudes (on isobaric surfaces)

This approach is the foundation for many other methods of analyzing atmospheric motions.

Scale Analysis

Relative Vorticity: $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \approx \frac{U}{L} \approx 10^{-5}\ \mathrm{s}^{-1}$

Planetary Vorticity: $f_0 \approx 10^{-4}\ \mathrm{s}^{-1}$

Definition: The **Rossby number** of a flow is a dimensionless quantity which represents the ratio of inertia to Coriolis force.

$$\frac{\zeta}{f_0} \approx \frac{U}{f_0 L} \equiv Ro$$

$\Rightarrow$ $\frac{\zeta}{f_0} \approx 10^{-1}$

In the mid-latitudes planetary vorticity is generally larger than relative vorticity.

Dynamical Equations

Pressure Coordinates

Momentum Equation

$$\frac{D\mathbf{u}_h}{Dt} + f\mathbf{k} \times \mathbf{u}_h = -\nabla_p \Phi$$

Hydrostatic Relation

$$\frac{\partial \Phi}{\partial p} = -\frac{R_d T}{p}$$

Continuity Equation

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)_p + \frac{\partial \omega}{\partial p} = 0$$

Thermodynamic Equation

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} - S_p \omega = \frac{J}{c_p}$$

Ideal Gas Law

$$p = \rho R_d T$$

Material Derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial p}$$

Quasi-Geostrophic Theory

Approximation (A)
Separation of geostrophic and ageostrophic wind

Recall the separation of the real wind into geostrophic and ageostrophic components:

$$\mathbf{u} = \mathbf{u}_g + \mathbf{u}_a \qquad \mathbf{u}_g = \frac{1}{f_0}\mathbf{k} \times \nabla\Phi$$

We assume that advection is dominated by the geostrophic winds:

$$\Rightarrow \quad \frac{D}{Dt} \approx \frac{D_g}{Dt} \equiv \frac{\partial}{\partial t} + u_g \frac{\partial}{\partial x} + v_g \frac{\partial}{\partial y}$$

Quasi-Geostrophic Theory

Approximation (B)
The mid-latitude beta plane approximation

Expand the Coriolis parameter as a Taylor series about a fixed latitude:

$$f = f_0 + \beta y$$

with $\quad f_0 = 2\Omega \sin \phi_0 \qquad \beta = \frac{2\Omega \cos \phi_0}{a}$

Using the Cartesian coordinate $\quad y = a(\phi - \phi_0)$

with a defined as the radius of the Earth

Note: A constant f_0 is still used for computing the geostrophic wind.

Quasi-Geostrophic Theory

Approximation (B)
The mid-latitude beta plane approximation

From the Coriolis parameter expansion:

$$f = f_0 + \beta y$$

and scale analysis for large-scale midlatitudinal systems:

$$\frac{\beta L}{f_0} \approx \frac{L \cos \phi_0}{a \sin \phi_0} \approx Ro \approx 10^{-1}$$

The second term in the beta-plane approximation is approximately an order of magnitude smaller than the first term.

Quasi-Geostrophic Theory

Approximation (C)
Small vertical temperature perturbations

Expand the total temperature in terms of a background temperature and a perturbation:

$$T(x, y, p, t) = T_0(p) + T'(x, y, p, t)$$

with $$\left|\frac{\partial T_0}{\partial p}\right| \gg \left|\frac{\partial T'}{\partial p}\right|$$

QG Momentum Equation

Starting from here:

$$\frac{D\mathbf{u}_h}{Dt} + f\mathbf{k} \times \mathbf{u}_h = -\nabla_p \Phi$$

Definition of geostrophic wind

$$\frac{D\mathbf{u}_h}{Dt} = -f\mathbf{k} \times \mathbf{u}_h - f_0\mathbf{k} \times \mathbf{u}_g$$

Using $f = f_0 + \beta y$ $\quad$ $\mathbf{u} = \mathbf{u}_g + \mathbf{u}_a$

$$\frac{D\mathbf{u}_h}{Dt} = -(f_0 + \beta y)\mathbf{k} \times (\mathbf{u}_g + \mathbf{u}_a) - f_0\mathbf{k} \times \mathbf{u}_g$$

QG Momentum Equation

$$\frac{D\mathbf{u}_h}{Dt} = -(f_0 + \beta y)\mathbf{k} \times (\mathbf{u}_g + \mathbf{u}_a) - f_0\mathbf{k} \times \mathbf{u}_g$$

Expand and Cancel

$$\frac{D\mathbf{u}_h}{Dt} = -f_0\mathbf{k} \times \mathbf{u}_a - \beta y\mathbf{k} \times \mathbf{u}_g - \beta y\mathbf{k} \times \mathbf{u}_a$$

Scales: 10^{-4} m/s^2 | 10^{-4} m/s^2 | 10^{-5} m/s^2

Observe: this term is much smaller than the others

$$\frac{D\mathbf{u}_h}{Dt} = -f_0\mathbf{k} \times \mathbf{u}_a - \beta y\mathbf{k} \times \mathbf{u}_g - \beta y\mathbf{k} \times \mathbf{u}_a$$

QG Momentum Equation

$$\frac{D\mathbf{u}_h}{Dt} = -f_0\mathbf{k} \times \mathbf{u}_a - \beta y \mathbf{k} \times \mathbf{u}_g$$

$$\frac{D\mathbf{u}}{Dt} \approx \frac{D_g\mathbf{u}_g}{Dt}$$

QG Momentum Equation

$$\frac{D_g\mathbf{u}_g}{Dt} = -f_0\mathbf{k} \times \mathbf{u}_a - \beta y \mathbf{k} \times \mathbf{u}_g$$

QG Momentum Equation

$$\frac{D\mathbf{u}_g}{Dt} = -f_0\mathbf{k} \times \mathbf{u}_a - \beta y\mathbf{k} \times \mathbf{u}_g$$

This equation states that the time rate of change of the geostrophic wind is related to:

1. The Coriolis force due to the ageostrophic wind.
2. The variation of the Coriolis force with latitude (β) multiplied by the geostrophic wind.

Both of these terms are smaller than the geostrophic wind itself.

QG Continuity Equation

Starting from here:

$$\nabla_p \cdot \mathbf{u} + \frac{\partial \omega}{\partial p} = 0$$

Geostrophic wind is non-divergent on pressure surfaces

$$\nabla_p \cdot \mathbf{u} = \nabla_p \cdot \mathbf{u}_a$$

QG Continuity Equation

$$\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} + \frac{\partial \omega}{\partial p} = 0$$

QG Thermodynamic Eq'n

Starting from here:

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} - S_p\omega = \frac{J}{c_p}$$

$$T(x,y,p,t) = T_0(p) + T'(x,y,p,t)$$

$$\left(\frac{\partial}{\partial t} + \mathbf{u}_g \cdot \nabla\right) T - \left(\frac{\sigma p}{R_d}\right)\omega = \frac{J}{c_p}$$

with $\sigma \equiv -\frac{R_d T_0}{p}\frac{\partial \ln \theta_0}{\partial p}$

QG Thermodynamic Eq'n

$$\left(\frac{\partial}{\partial t} + \mathbf{u}_g \cdot \nabla\right) T - \left(\frac{\sigma p}{R_d}\right) \omega = \frac{J}{c_p}$$

Hydrostatic Relation

$$\frac{\partial \Phi}{\partial p} = -\frac{R_d T}{p}$$

QG Thermodynamic Equation

$$\left(\frac{\partial}{\partial t} + \mathbf{u}_g \cdot \nabla\right) \left(-\frac{\partial \Phi}{\partial p}\right) - \sigma \omega = \frac{\kappa J}{p}$$

with $\kappa \equiv \frac{R_d}{c_p}$ $\quad \sigma \equiv -\frac{R_d T_0}{p} \frac{\partial \ln \theta_0}{\partial p}$

QG Equations

Geostrophic Wind

$$\mathbf{u}_g = \frac{1}{f_0}\mathbf{k} \times \nabla\Phi$$

Momentum Equation

$$\frac{D_g \mathbf{u}_g}{Dt} = -f_0 \mathbf{k} \times \mathbf{u}_a - \beta y \mathbf{k} \times \mathbf{u}_g$$

Continuity Equation

$$\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} + \frac{\partial \omega}{\partial p} = 0$$

Thermodynamic Equation

$$\left(\frac{\partial}{\partial t} + \mathbf{u}_g \cdot \nabla\right)\left(-\frac{\partial \Phi}{\partial p}\right) - \sigma\omega = \frac{\kappa J}{p}$$

Auxiliary Equations

$$\kappa \equiv \frac{R_d}{c_p} \qquad \sigma \equiv -\frac{R_d T_0}{p}\frac{\partial \ln \theta_0}{\partial p}$$

→ $\Phi, \mathbf{u}_g, \mathbf{u}_a, \omega$ are independent variables, form a complete set if heating rate *J* is known

QG Equations

In deriving the QG equations we used scale analysis. That is, we made assumptions on the scales of the phenomena we are studying.

Quasi-geostrophic system is ***good*** for:

- Synoptic scales
- Middle latitudes
- Situations in which $\mathbf{u_a}$ is important
- Flows in approximate geostrophic and hydrostatic balance
- ***Mid-latitude cyclones***

QG Equations

In deriving the QG equations we used scale analysis. That is, we made assumptions on the scales of the phenomena we are studying.

Quasi-geostrophic system is ***not good*** for:

- Very small or very large scales
- Flows with large vertical velocities
- Situations in which $\mathbf{u_a} \approx \mathbf{u_g}$
- Flows ***not*** in approximate geostrophic and hydrostatic balance
- ***Thunderstorms/convection, boundary layer, tropics, etc…***

QG Equations

Question: What's the point?

A set of equations that describes synoptic-scale motions and ***includes the effects of ageostrophic wind*** (vertical motion).

This moves us towards a set of simple predictive equations for atmospheric motions in the mid-latitudes.

QG Equations

Question: What's the point?

We want to describe the evolution of two key features of the atmosphere:

- **Large-scale waves** (in particular, the connection between large-scale waves and geopotential)

- **Midlatitude cyclones** (that is, the development of low pressure systems in the lower troposphere)

QG Vorticity Equation

The next goal is to derive a vorticity equation for these scaled equations.

- This equation actually provides a "suitable" prognostic equation because will include the divergence of the ageostrophic wind (and hence account for vertical motion).

- Recall that divergence is a dominant mechanism for the generation of vorticity in the vorticity equation…

QG Vorticity Equation

QG Momentum Equation

$$\frac{D_g \mathbf{u}_g}{Dt} = -f_0 \mathbf{k} \times \mathbf{u}_a - \beta y \mathbf{k} \times \mathbf{u}_g$$

Component Form

$$\frac{D_g u_g}{Dt} - f_0 v_a - \beta y v_g = 0$$

$$\frac{D_g v_g}{Dt} + f_0 u_a + \beta y u_g = 0$$

Then compute $$\frac{\partial}{\partial x}\left(\frac{D_g v_g}{Dt}\right) - \frac{\partial}{\partial y}\left(\frac{D_g u_g}{Dt}\right)$$

QG Vorticity Equation

$$\frac{D_g \zeta_g}{Dt} = -f_0 \left(\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} \right) - \beta v_g$$

Expand D_g/Dt

$$\frac{\partial \zeta_g}{\partial t} + \mathbf{u}_g \cdot \nabla \zeta_g = -f_0 \left(\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} \right) - \beta v_g$$

$$\frac{\partial \zeta_g}{\partial t} = f_0 \frac{\partial \omega}{\partial p} - \mathbf{u}_g \cdot \nabla \zeta_g - \beta v_g$$

These forms of the QG vorticity equation are all equivalent

QG Vorticity Equation

$$\frac{\partial \zeta_g}{\partial t} = f_0 \frac{\partial \omega}{\partial p} - \mathbf{u}_g \cdot \nabla (\zeta_g + f)$$

QG Vorticity Equation

Scaled Vorticity Equation

$$\frac{D_h \zeta}{Dt} = -(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - v\frac{\partial f}{\partial y}$$

QG Vorticity Equation

$$\frac{D_g \zeta_g}{Dt} = -f_0\left(\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y}\right) - \beta v_g$$

The QG vorticity equation is very similar to the scaled vorticity equation we developed earlier, with a few additional assumptions (highlighted).

QG Vorticity Equation

$$\frac{\partial \zeta_g}{\partial t} = f_0 \frac{\partial \omega}{\partial p} - \mathbf{u}_g \cdot \nabla(\zeta_g + f)$$

$$\frac{\partial \zeta_g}{\partial t} = f_0 \frac{\partial \omega}{\partial p} - \mathbf{u}_g \cdot \nabla \zeta_g - \beta v_g$$

Stretching Term

Advection of relative vorticity

Advection of planetary vorticity

Competing