A Rotational View of the Atmoshere

Chapter 4

Paul A. Ullrich
paullrich@ucdavis.edu

Part 3: Potential Vorticity

Barotropic Vorticity Equation

Assume a homogeneous, incompressible fluid
(constant density with $\varrho = 0$).

Continuity Equation

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{u})$$

$$\nabla \cdot \mathbf{u} = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial w}{\partial z}$$

Barotropic Vorticity Equation

Vorticity Equation
(Divergence term only)

$$\frac{D_h}{Dt}(\zeta + f) = -(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial w}{\partial z}$$

$$\frac{D_h(\zeta + f)}{Dt} = (\zeta + f)\frac{\partial w}{\partial z}$$

For purely horizontal flow (*w=0*):

$$\frac{D_h(\zeta + f)}{Dt} = 0$$

Barotropic Vorticity Equation

The barotropic vorticity equation states that when there is no vertical velocity, the **absolute vorticity** is **conserved** following the horizontal motion.

$$\frac{D_h(\zeta + f)}{Dt} = 0$$

Let's try a more general derivation…

Potential Vorticity

In a Barotropic Fluid

Assume a homogeneous, incompressible fluid (constant density with $\varrho = 0$).

$$\frac{D_h(\zeta + f)}{Dt} = (\zeta + f)\frac{\partial w}{\partial z}$$

Approximate vorticity by geostrophic vorticity, integrate vertically from z_1 to z_2

$$\frac{D_h(\zeta_g + f)}{Dt} = (\zeta_g + f)\frac{w(z_2) - w(z_1)}{z_2 - z_1}$$

Use $\frac{D_h h}{Dt} = w(z_2) - w(z_1)$ $\Rightarrow$ $h\frac{D_h(\zeta_g + f)}{Dt} = (\zeta_g + f)\frac{D_h h}{Dt}$

Potential Vorticity

In a Barotropic Fluid

$$\frac{D_h(\zeta_g + f)}{Dt} = (\zeta_g + f)\frac{w(z_2) - w(z_1)}{z_2 - z_1}$$

Define layer thickness

$$h = z_2 - z_1$$

Use $$\frac{D_h h}{Dt} = w(z_2) - w(z_1)$$

$$h\frac{D_h(\zeta_g + f)}{Dt} = (\zeta_g + f)\frac{D_h h}{Dt}$$

Rearrange

$$\frac{1}{(\zeta_g + f)}\frac{D_h(\zeta_g + f)}{Dt} = \frac{1}{h}\frac{D_h h}{Dt}$$

Potential Vorticity

In a Barotropic Fluid

$$\frac{1}{(\zeta_g + f)} \frac{D_h(\zeta_g + f)}{Dt} = \frac{1}{h} \frac{D_h h}{Dt}$$

Calculus of logarithms

$$\frac{D_h}{Dt} \log(\zeta_g + f) = \frac{D_h}{Dt} \log h$$

Properties of logarithms

$$\frac{D_h}{Dt} \log \frac{(\zeta_g + f)}{h} = 0$$

$$\boxed{\frac{D_h}{Dt} \left[\frac{\zeta_g + f}{h} \right] = 0}$$

Potential Vorticity

In a Barotropic Fluid

For a barotropic, incompressible and homogeneous fluid:

$$\frac{D_h}{Dt}\left[\frac{\zeta_g + f}{h}\right] = 0$$

Definition: The **barotropic potential vorticity** of a fluid column is defined as

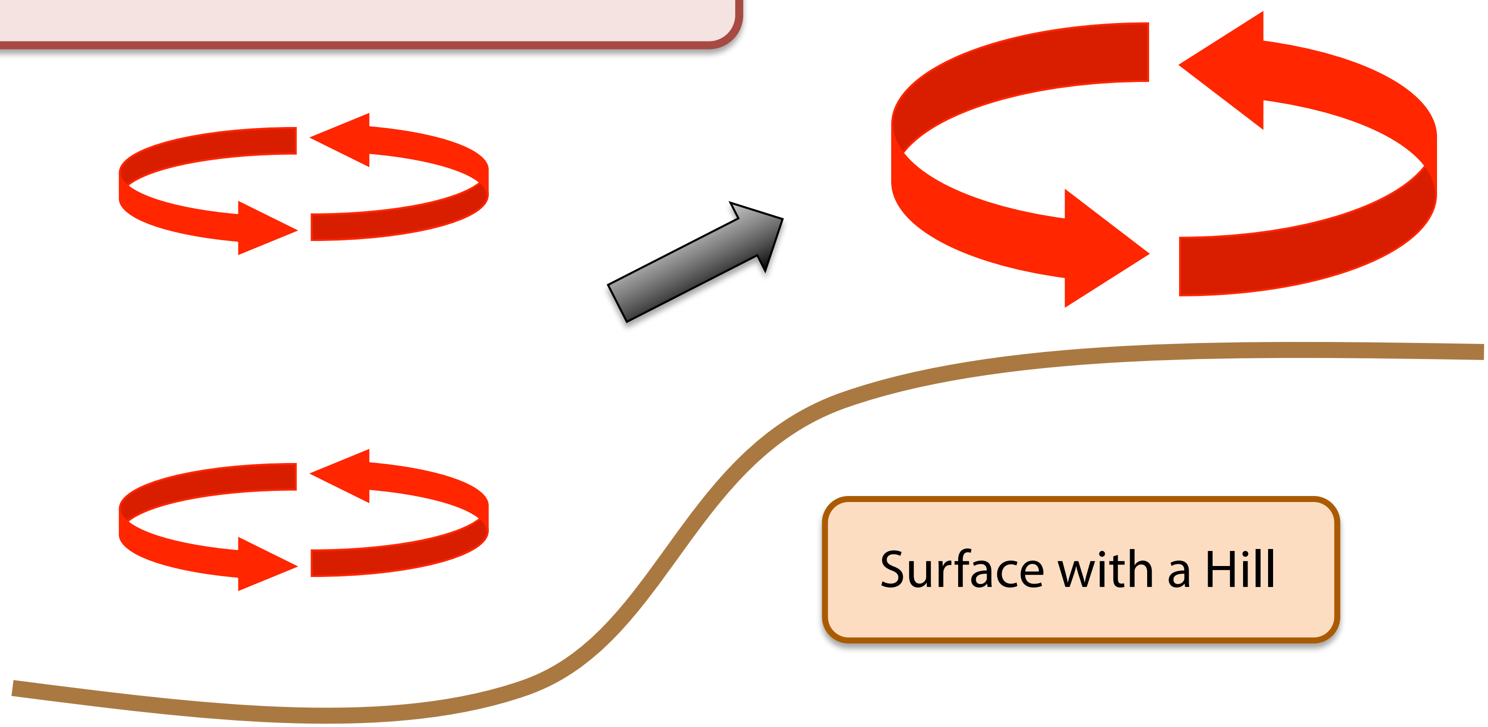
$$PV = \frac{\zeta_g + f}{h}$$

Barotropic potential vorticity is conserved following a fluid column and measures the absolute vorticity relative to the depth of the vortex.

PV Conservation

In a Barotropic Fluid

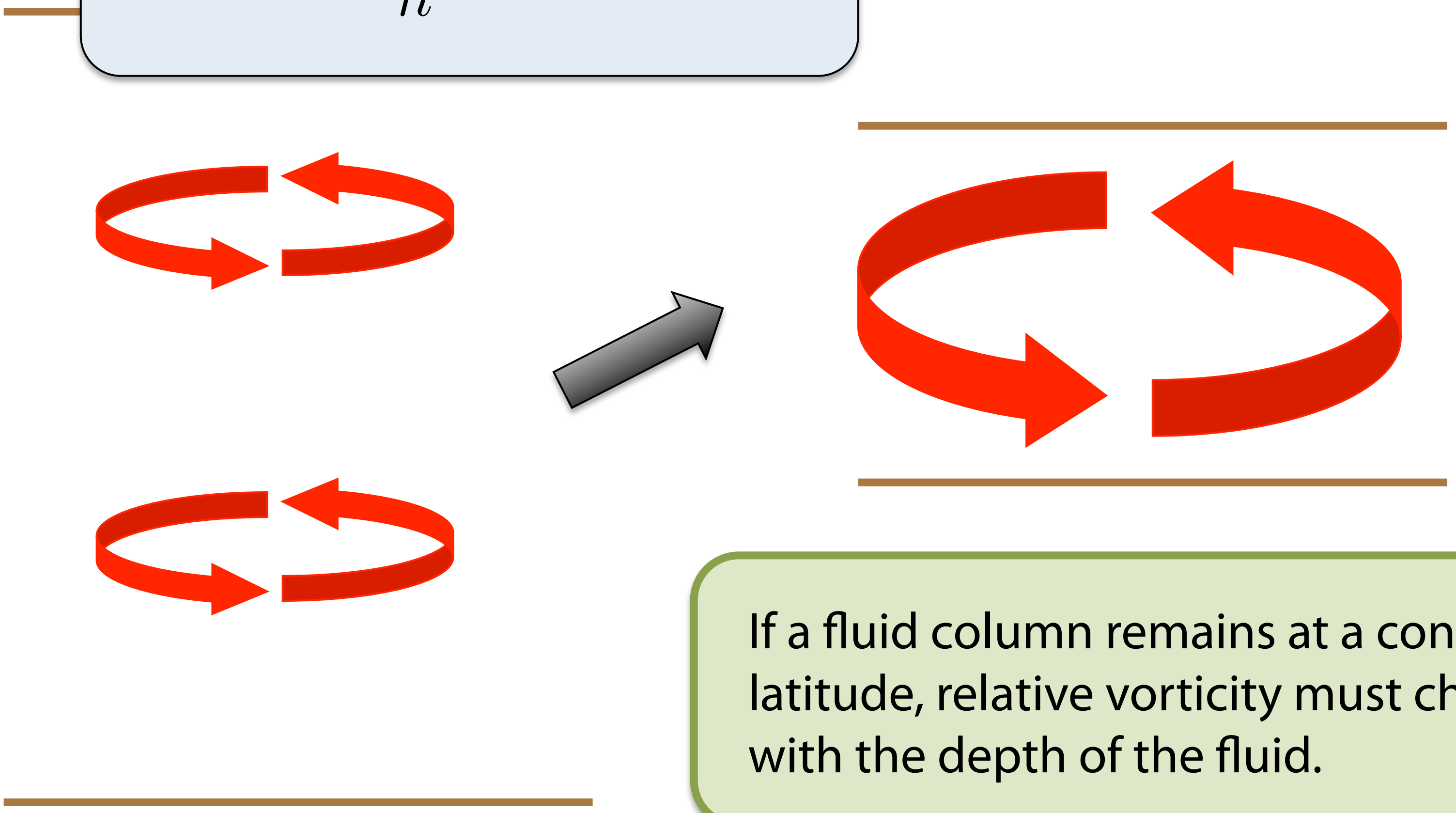
Question: What happens when the vortex meets a hill?



PV Conservation

In a Barotropic Fluid

$$PV = \frac{\zeta_g + f}{h} = \text{Constant}$$



If a fluid column remains at a constant latitude, relative vorticity must change with the depth of the fluid.

Vorticity and Depth

Potential vorticity provides a link between **depth** and **vorticity**. As the depth of the vortex changes, the relative vorticity has to change in order to conserve potential vorticity.

$$PV = \frac{\zeta_g + f}{h} = \text{Constant}$$

This result links:

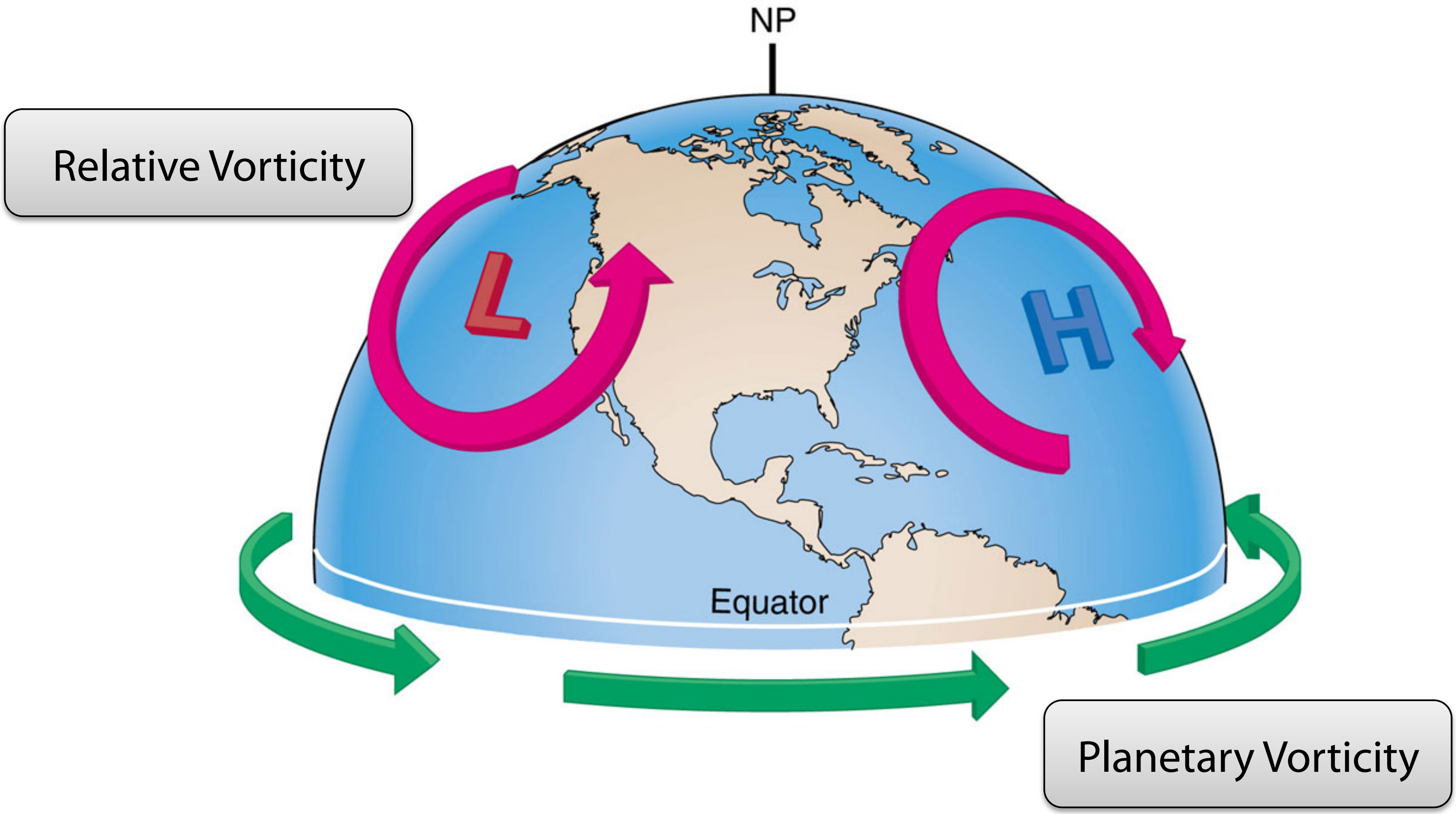
- The **rotational component** of the wind, which is responsible for rotation in the horizontal plane.
- The **irrotational component**, which is responsible for divergence / convergence and hence changes in the height of the fluid column.

Vorticity and Depth

Potential vorticity indicates an inerplay between relative and planetary vorticity through conservation of absolute angular momentum.

$$PV = \frac{\zeta_g + f}{h} = \text{Constant}$$

Vorticity



The Vorticity Equation

$$\frac{\partial \zeta}{\partial t} \boxed{+ \mathbf{u} \cdot \nabla \zeta} \boxed{+ (\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)}$$

$$\boxed{+ \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)} \boxed{+ v\frac{\partial f}{\partial y}} = \boxed{-\frac{\partial}{\partial x}\left(\frac{1}{\rho}\right)\frac{\partial p}{\partial y} + \frac{\partial}{\partial y}\left(\frac{1}{\rho}\right)\frac{\partial p}{\partial x}}$$

Changes in absolute vorticity are caused by:

Advection

Divergence

Tilting

Baroclinicity

Scale Analysis

Relative Vorticity: $$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \approx \frac{U}{L} \approx 10^{-5}\ \mathrm{s}^{-1}$$

Planetary Vorticity: $$f_0 \approx 10^{-4}\ \mathrm{s}^{-1}$$

$$\Rightarrow \quad \frac{\zeta}{f_0} \approx 10^{-1}$$

In the mid-latitudes planetary vorticity is generally larger than relative vorticity.

Relative / Planetary Vorticity

Observation 1: Planetary vorticity is cyclonic / positive vorticity in the Northern Hemisphere.

Observation 2: Planetary vorticity, in the mid-latitudes, is usually larger than relative vorticity.

⇒ A growing cyclone "adds to" planetary vorticity, leading to intense lows.

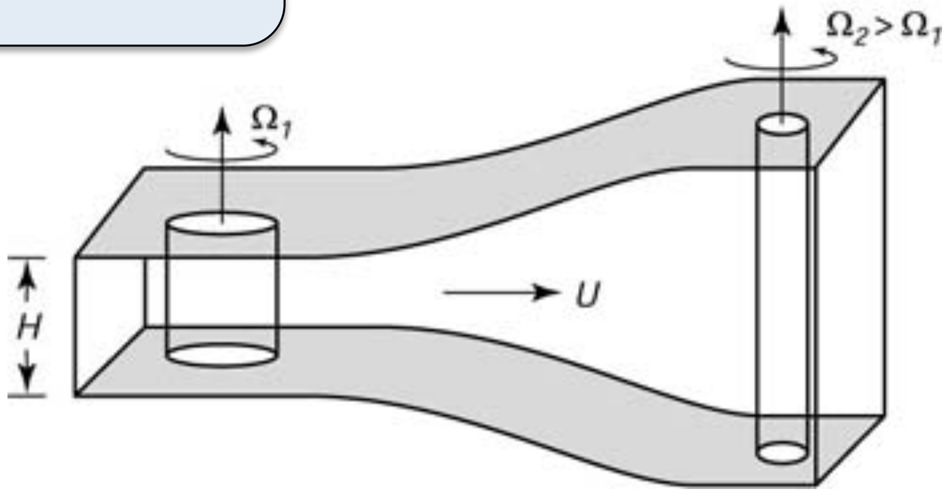
⇒ A growing anti-cyclone "opposes" planetary vorticity, leading to weak highs.

PV Conservation

In a Barotropic Fluid

$$PV = \frac{\zeta_g + f}{h} = \text{Constant}$$

Figure: Stretching and shrinking of a column will change the relative vorticity.



Recall: Constant density implies a fluid does not change in volume.