A Rotational View of the Atmoshere

Chapter 4

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Part 2: The Vorticity Equation

Vorticity: Key Questions

Question: How is positive / negative vorticity generated?

Question: How do we describe the time rate of change of vorticity?

Question: How do we describe conservation of vorticity? Is vorticity actually conserved following the flow?

Question: What is the role of the Earth's rotation?

The Vorticity Equation

Start from the scaled horizontal momentum equation in z coordinates, no viscosity:

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv$$
$$\frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu$$

Take derivatives on both sides and compute:

$$\frac{\partial}{\partial x}\left(\frac{Dv}{Dt}\right) - \frac{\partial}{\partial y}\left(\frac{Du}{Dt}\right) \quad \left[\begin{array}{l} \dfrac{\partial}{\partial x}\left(\dfrac{Dv}{Dt}\right) = \dfrac{\partial}{\partial x}\left(\dfrac{\partial v}{\partial t} + \mathbf{u}\cdot\nabla v\right) \\ \dfrac{\partial}{\partial y}\left(\dfrac{Du}{Dt}\right) = \dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial t} + \mathbf{u}\cdot\nabla u\right) \end{array} \right.$$

The Vorticity Equation

$$-\frac{\partial}{\partial y}\frac{\partial u}{\partial t} - \frac{\partial}{\partial y}(\mathbf{u}\cdot\nabla u) = +\frac{\partial}{\partial y}\left(\frac{1}{\rho}\frac{\partial p}{\partial x}\right) - \frac{\partial}{\partial y}(fv)$$

$$+\frac{\partial}{\partial x}\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}(\mathbf{u}\cdot\nabla v) = -\frac{\partial}{\partial x}\left(\frac{1}{\rho}\frac{\partial p}{\partial y}\right) - \frac{\partial}{\partial x}(fu)$$

A rather long derivation

$$\frac{\partial \zeta}{\partial t} + \mathbf{u}\cdot\nabla\zeta + (\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

$$+\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + v\frac{\partial f}{\partial y} = -\frac{\partial}{\partial x}\left(\frac{1}{\rho}\right)\frac{\partial p}{\partial y} + \frac{\partial}{\partial y}\left(\frac{1}{\rho}\right)\frac{\partial p}{\partial x}$$

The Vorticity Equation

$$\frac{\partial \zeta}{\partial t} + \mathbf{u} \cdot \nabla \zeta + (\zeta + f) \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)$$
$$+ \left(\frac{\partial w}{\partial x} \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} \frac{\partial u}{\partial z} \right) + v \frac{\partial f}{\partial y} = -\frac{\partial}{\partial x} \left(\frac{1}{\rho} \right) \frac{\partial p}{\partial y} + \frac{\partial}{\partial y} \left(\frac{1}{\rho} \right) \frac{\partial p}{\partial x}$$

Vorticity Equation

$$\frac{D}{Dt}(\zeta + f) = -(\zeta + f)(\nabla_h \cdot \mathbf{u}) - \left(\frac{\partial w}{\partial x} \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} \frac{\partial u}{\partial z} \right) - (\nabla \alpha \times \nabla p) \cdot \mathbf{k}$$

With specific volume $\alpha = \frac{1}{\rho}$

The Vorticity Equation

What are these terms?

$$\frac{D}{Dt}(\zeta + f) = -(\zeta + f)(\nabla_h \cdot \mathbf{u}) - \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z} \right) - (\nabla \alpha \times \nabla p) \cdot \mathbf{k}$$

Divergence term

Tilting or twisting term

Solenoidal term

Rate of change of the absolute vorticity following the motion

The Vorticity Equation

Take a closer look at the divergence term…

$$-(\zeta + f)(\nabla_h \cdot \mathbf{u})$$

Absolute vorticity

Horizontal divergence

- Remember that divergence is related to vertical motion (column integrated divergence gives the local vertical velocity)

- We now see that it is also related to changes in vorticity…

Vertical Velocity

Obtained from horizontal divergence

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)_p + \frac{\partial \omega}{\partial p} = 0$$

Continuity Equation in Pressure Coordinates

$$d\omega = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)_p dp$$

$$\int_{\omega\,@\,p_s}^{\omega\,@\,p=0} d\omega = -\int_{p_s}^{p=0} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)_p dp$$

$$\omega(p=0) - \omega(p_s) = -\int_{p_s}^{p=0} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)_p dp$$

$$\omega(p_s) = -\int_{p=0}^{p_s} (\nabla_p \cdot \mathbf{u}) dp$$

Vorticity and Divergence

Vertical wind is related to the divergence of the horizontal wind.

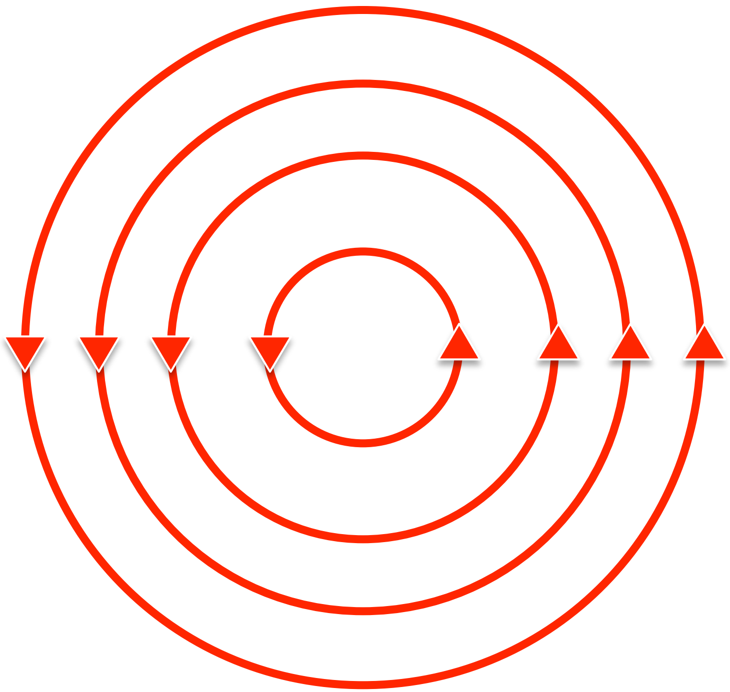
(which requires an ageostrophic component to the wind)

⇨

Changes in vorticity are partially driven by divergence of the horizontal wind.

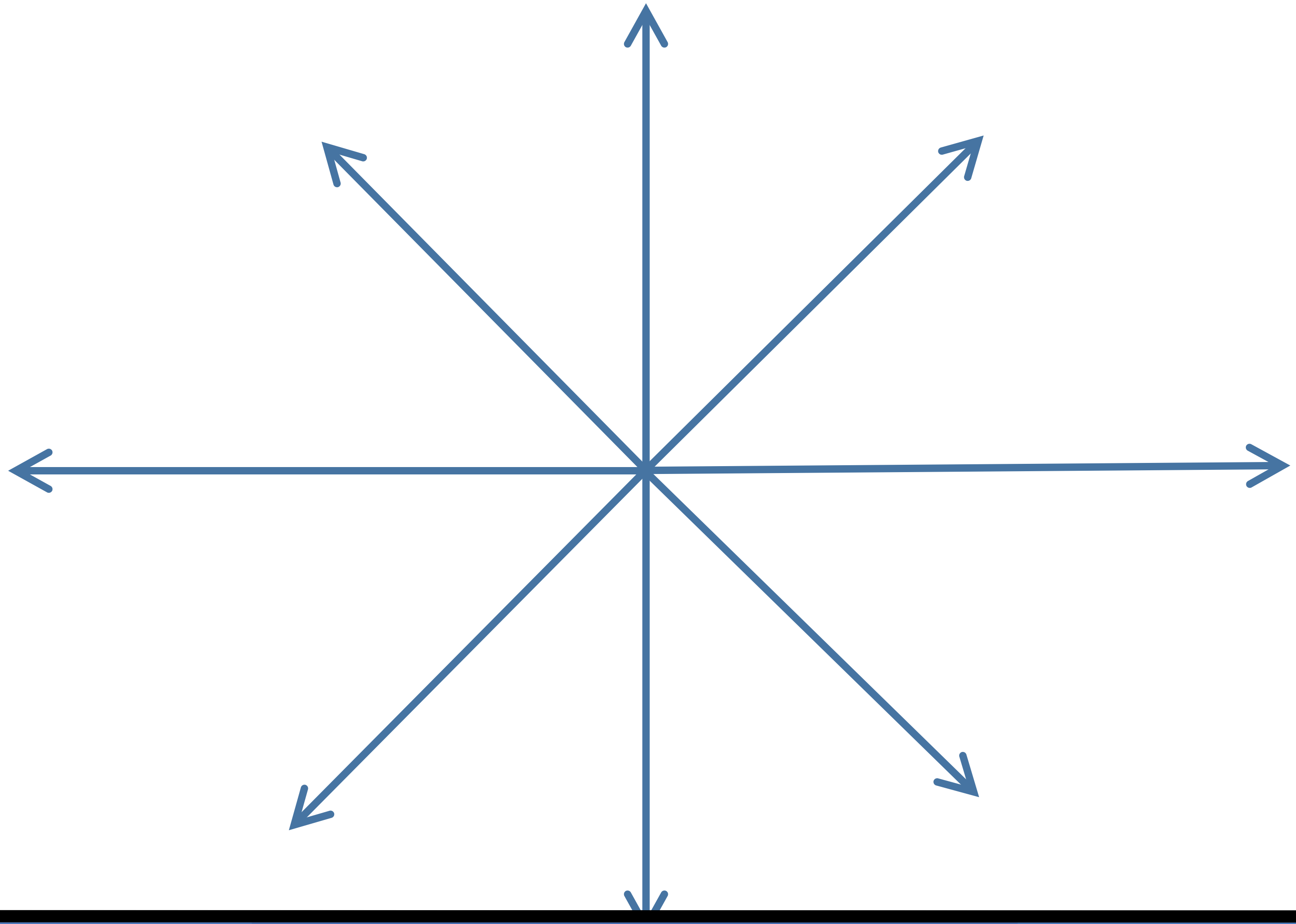
Vorticity and Divergence

This flow field is purely vortical (zero divergence)



Vorticity and Divergence

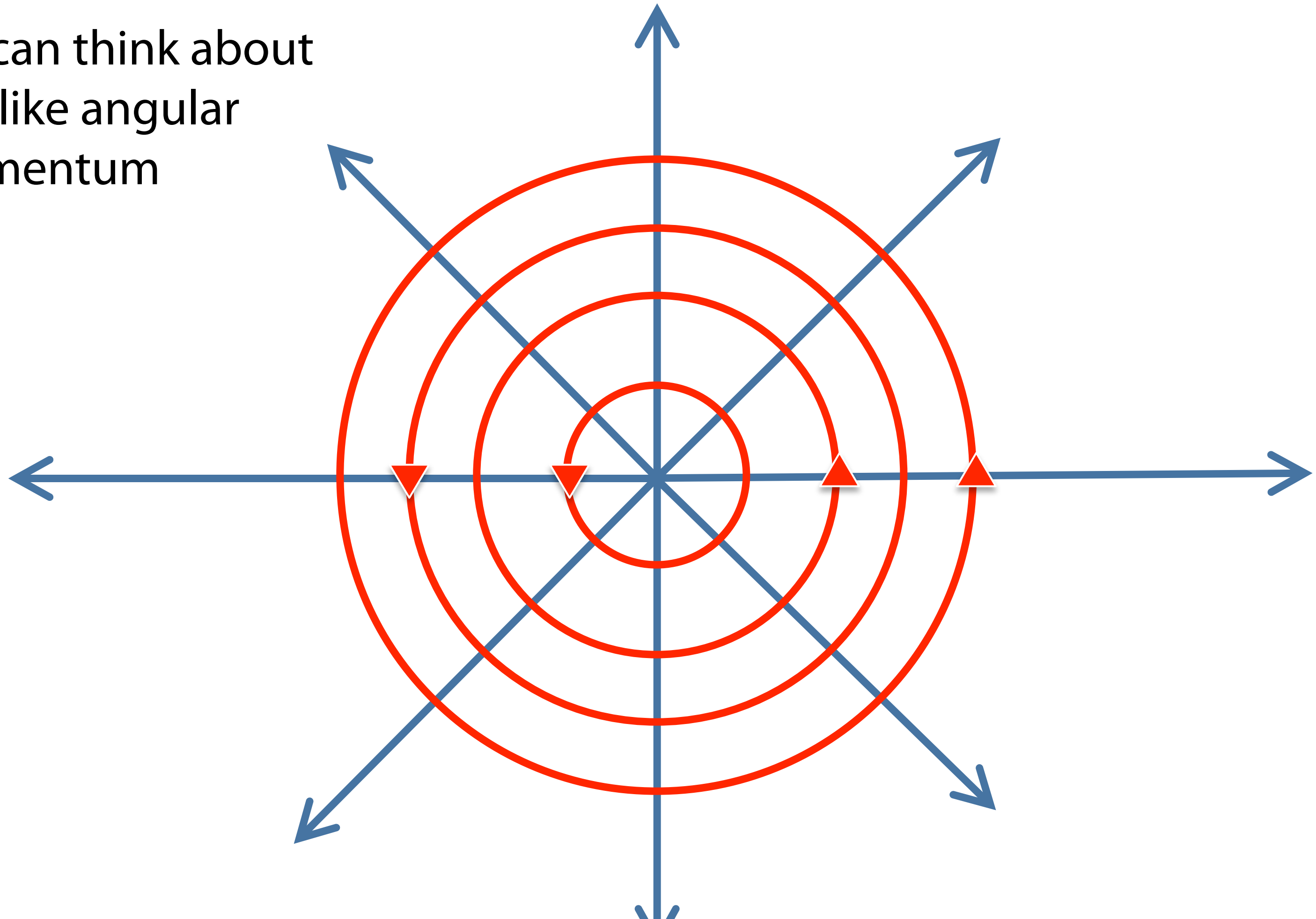
This flow field is purely divergent (zero vorticity)



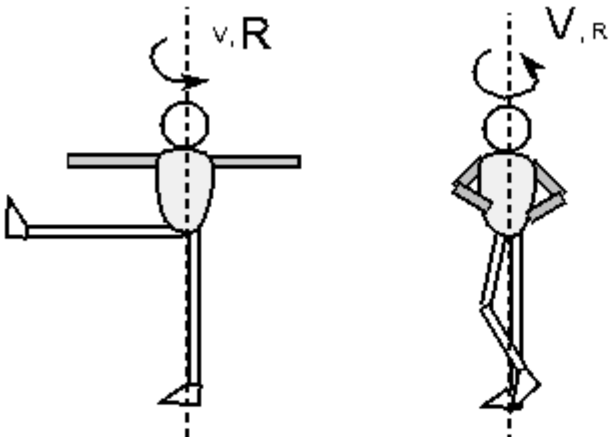
Vorticity and Divergence

Question: What is the effect of divergence on vorticity?

We can think about this like angular momentum



A Spinning Skater



Motion is in the (x,y) plane

Axis of rotation is in the vertical plane

A Spinning Skater

Question: A skater that is spinning with a given radius then brings in her arms. What happens to her angular velocity?

Motion is in the (x,y) plane

A Spinning Skater

Recall: Angular momentum is a conserved quantity. Its magnitude is given by

$$|\mathbf{L}| = r\, m\, \omega$$

Motion is in the (x,y) plane

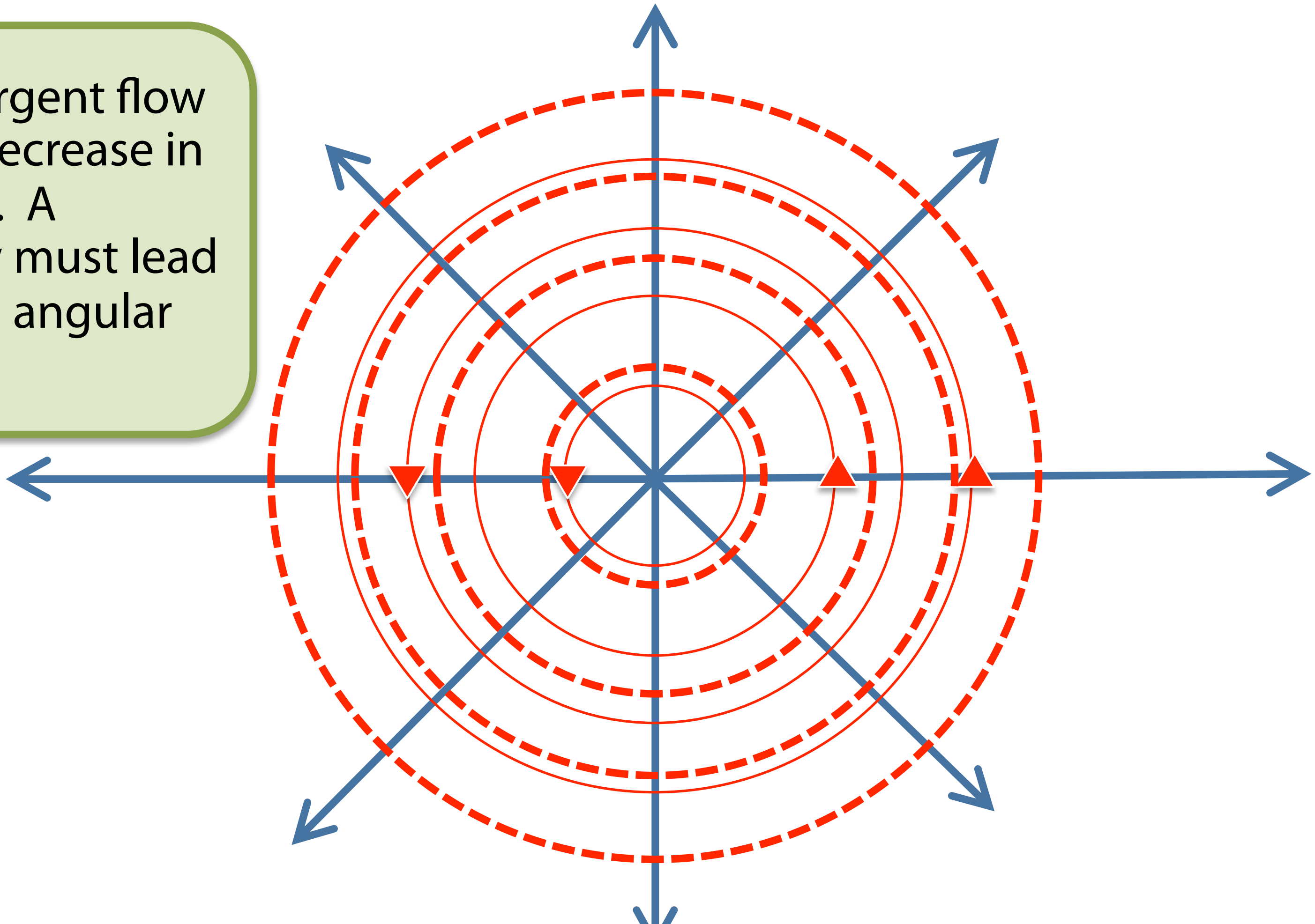
Answer: Since her radius decreases, conservation of angular momentum says that her angular velocity must increase.

Question: How is this like divergence?

Vorticity and Divergence

Question: What is the effect of divergence on vorticity?

Answer: A divergent flow must lead to a decrease in angular velocity. A convergent flow must lead to an increase in angular velocity.



The Vorticity Equation

Divergence term only

$$\frac{D}{Dt}(\zeta + f) = -(\zeta + f)\,(\nabla_h \cdot \mathbf{u})$$

Absolute vorticity

Horizontal divergence

Question: What happens to absolute vorticity when the flow is

(a) Divergent? $(\nabla_h \cdot \mathbf{u}) > 0$

(b) Convergent? $(\nabla_h \cdot \mathbf{u}) < 0$

The Vorticity Equation

What are these terms?

$$\frac{D}{Dt}(\zeta + f) = -(\zeta + f)(\nabla_h \cdot \mathbf{u}) - \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z} \right) - (\nabla\alpha \times \nabla p) \cdot \mathbf{k}$$

Divergence term

Tilting or twisting term

Solenoidal term

Rate of change of the absolute vorticity following the motion

Tilting and Twisting

Recall how relative vorticity is obtained:

Relative velocity: $$\mathbf{u} = (u, v, w)$$

3D vorticity vector: $$\nabla \times \mathbf{u} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$

Relative vorticity: $$\zeta = \mathbf{k} \cdot (\nabla \times \mathbf{u}) = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$

Tilting and Twisting

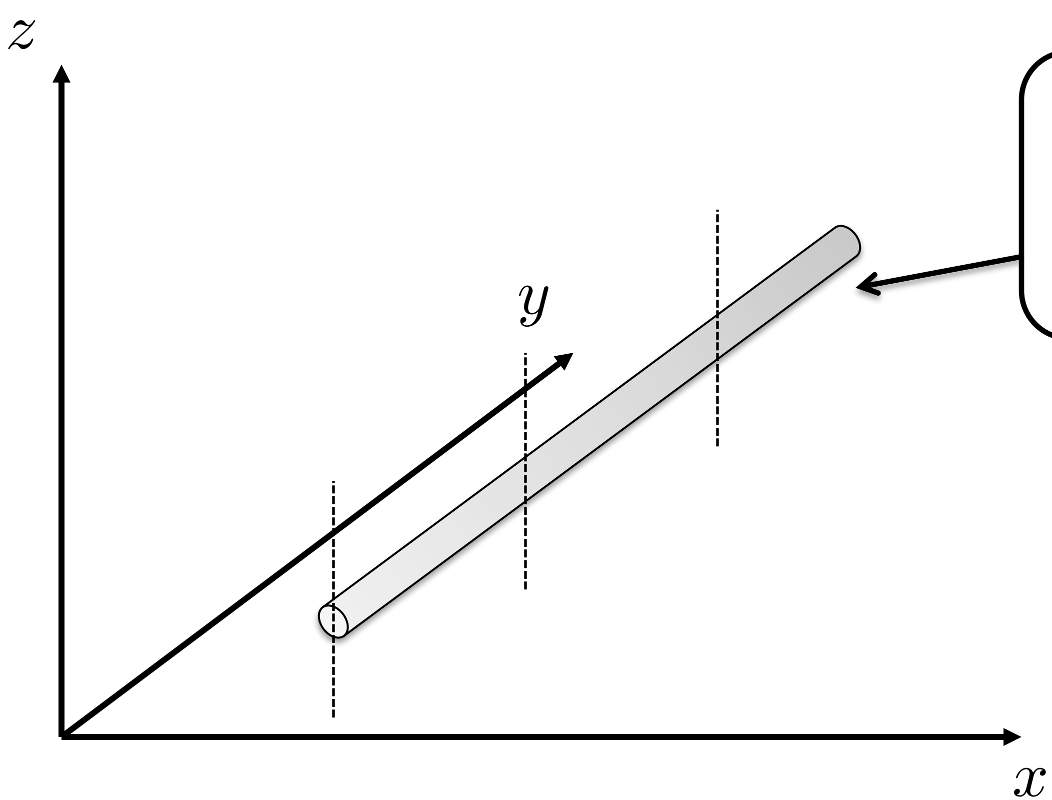
Tilting or twisting term

$$\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)$$

Tilting represents the change in the vertical component of vorticity by tilting horizontal vorticity into the vertical.

Tilting or twisting term

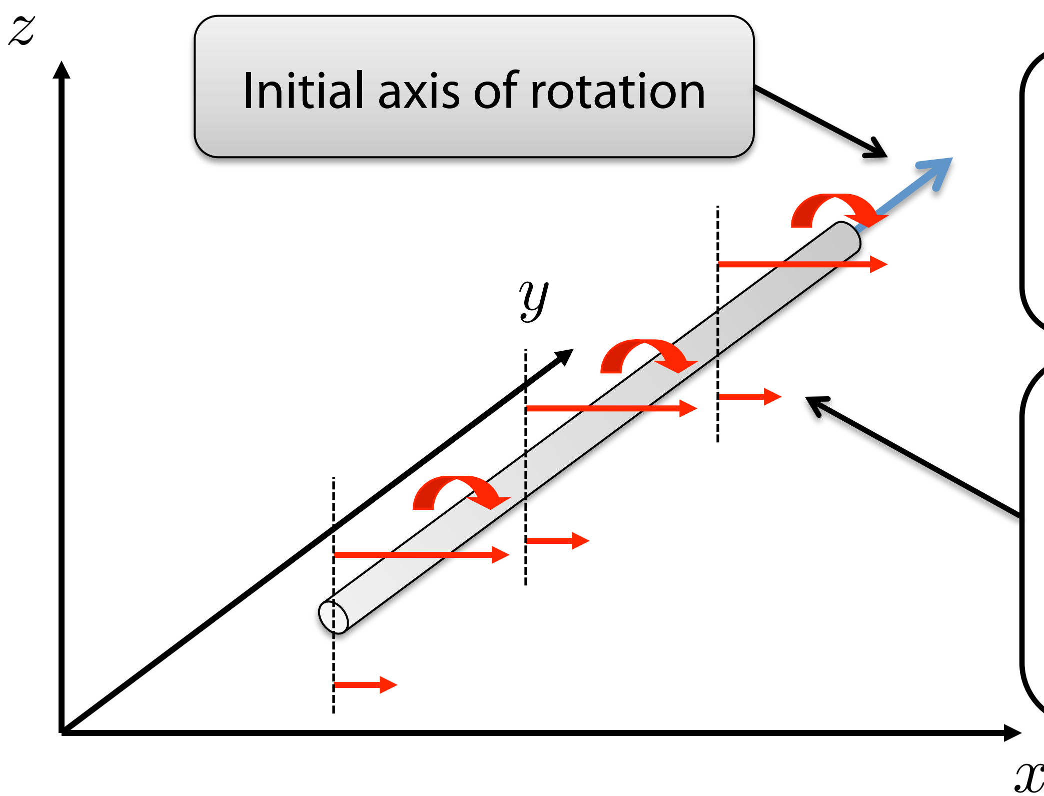
$$\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)$$



Consider a long, thin cylinder of fluid aligned with the y-axis.

Tilting or twisting term

$$\left(\underbrace{\frac{\partial w}{\partial x}\cancel{\frac{\partial v}{\partial z}}}_{=0} - \underbrace{\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}}_{>0}\right)$$



Consider a long, thin cylinder of fluid aligned with the y-axis.

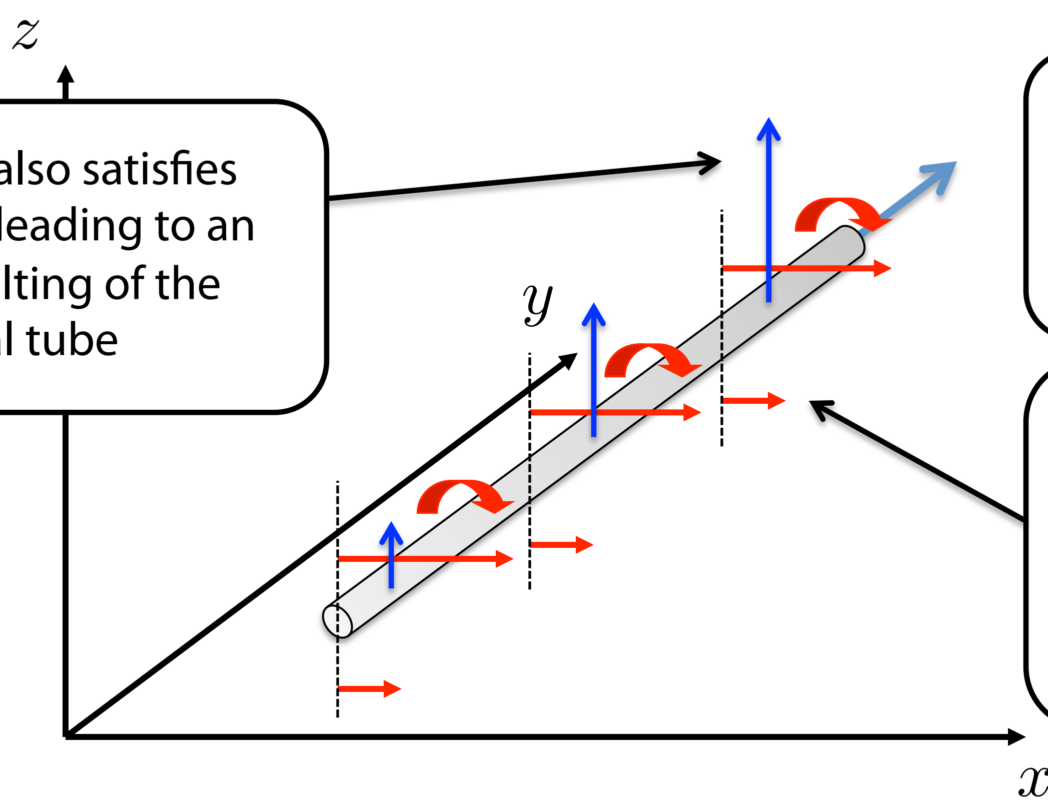
The flow satisfies $v=0$ and is sheared in the vertical so that the vorticity vector is along the y axis.

Tilting or twisting term

$$\left(\frac{\partial w}{\partial x}\cancel{\frac{\partial v}{\partial z}} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)$$

$= 0 \qquad > 0 \qquad > 0$

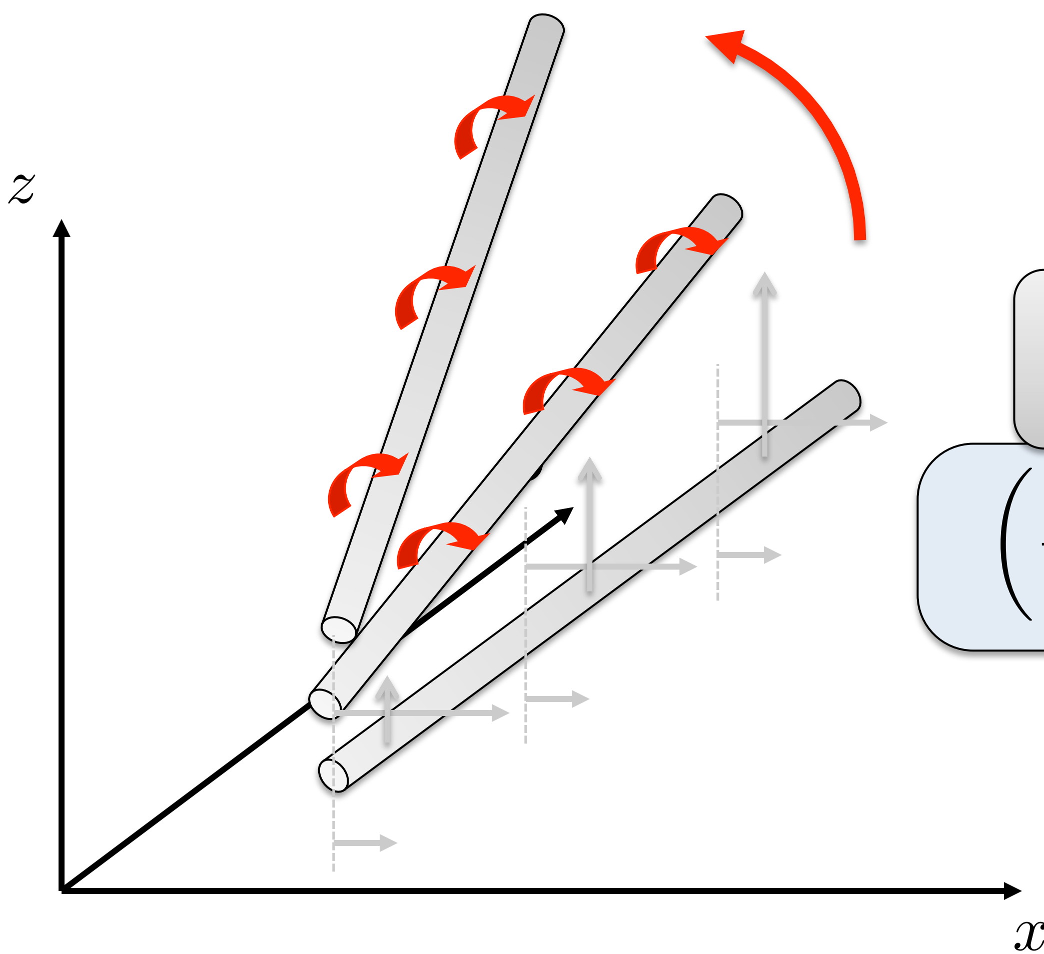
The flow also satisfies *dw/dy>0* leading to an upward tilting of the cylindrical tube



Consider a long, thin cylinder of fluid aligned with the y-axis.

The flow satisfies *v=0* and is sheared in the vertical so that the vorticity vector is along the y axis.

Tilting and Twisting



Tilting or twisting term

$$\left(\underbrace{\cancel{\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}}}_{=0} - \underbrace{\frac{\partial w}{\partial y}}_{>0}\underbrace{\frac{\partial u}{\partial z}}_{>0}\right)$$

Tilting and Twisting

Rotation in the (y,z) plane. Vorticity vector points along the x axis.

As the wheel is turned there is a component of vorticity in the z plane.

The Vorticity Equation

What are these terms?

$$\frac{D}{Dt}(\zeta + f) = -(\zeta + f)(\nabla_h \cdot \mathbf{u}) - \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z} \right) - (\nabla \alpha \times \nabla p) \cdot \mathbf{k}$$

Divergence term

Tilting or twisting term

Solenoidal term

Rate of change of the absolute vorticity following the motion

Solenoidal / Baroclinic Terms

$$-\frac{\partial}{\partial x}\left(\frac{1}{\rho}\frac{\partial p}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{1}{\rho}\frac{\partial p}{\partial x}\right) = -\frac{\partial \alpha}{\partial x}\frac{\partial p}{\partial y} + \frac{\partial \alpha}{\partial y}\frac{\partial p}{\partial x}$$

The solenoidal / baroclinic terms capture changes in vorticity due to the alignment of surfaces of constant density and pressure.

Solenoidal / Baroclinic Terms

Definition: In a **barotropic fluid** density depends only on pressure.

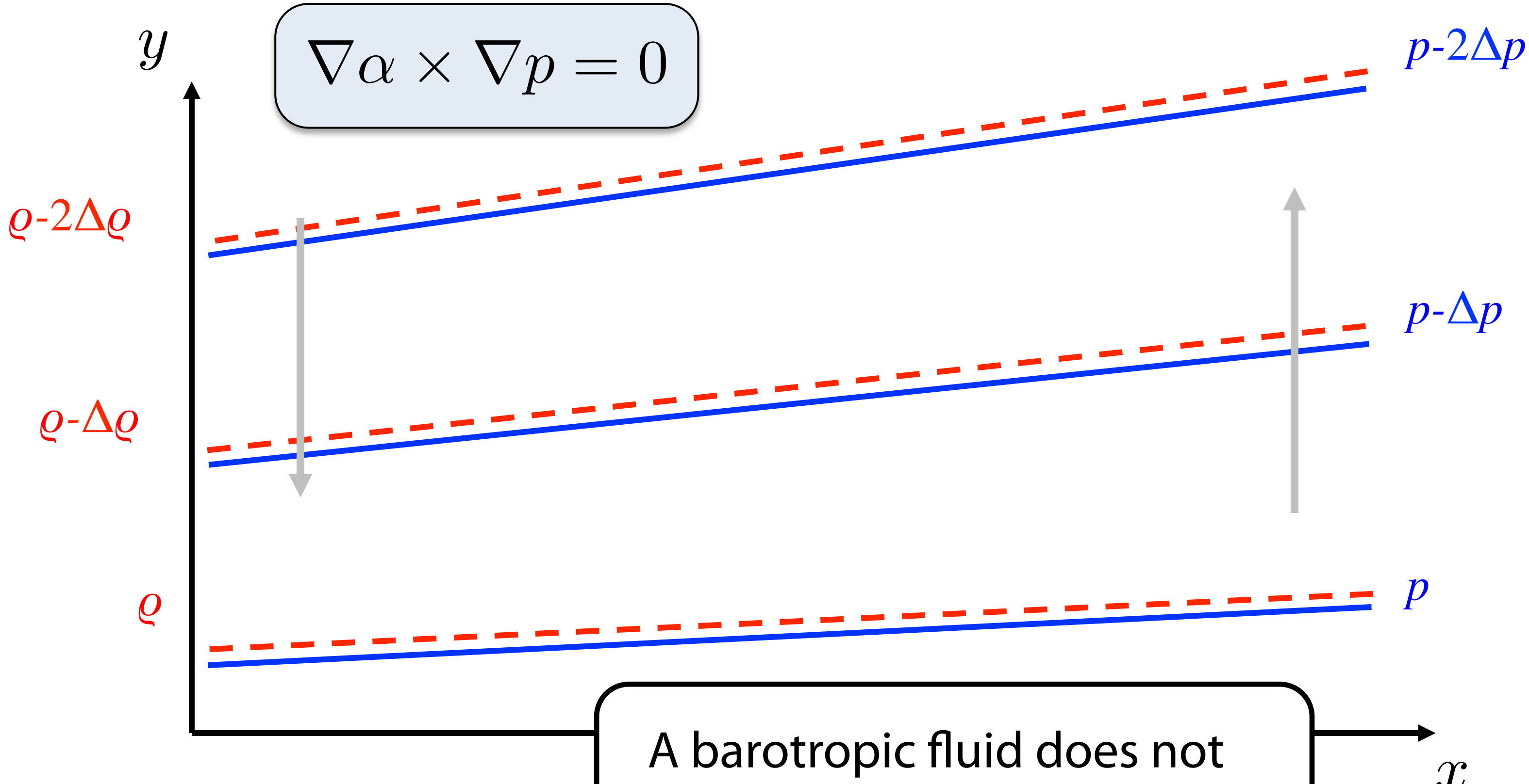
By the ideal gas law, this implies that surfaces of constant density are surfaces of constant pressure are surfaces of constant temperature.

Definition: In a **baroclinic fluid** density depends on pressure and temperature.

Solenoidal / Baroclinic Terms

Barotropic: surfaces of constant density parallel to surfaces of constant pressure

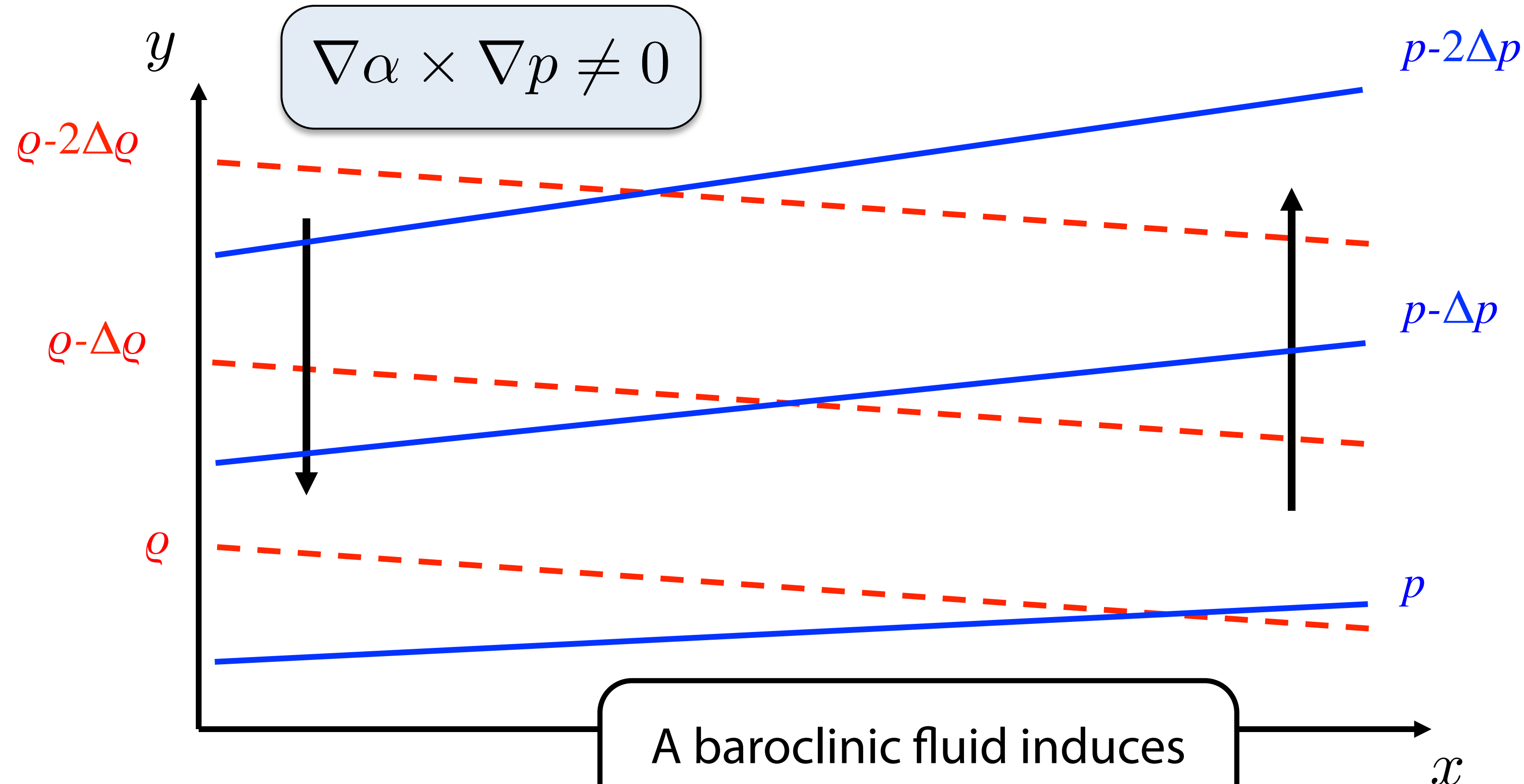
$$\nabla \alpha \times \nabla p = 0$$



A barotropic fluid does not induce changes in vorticity

Solenoidal / Baroclinic Terms

Baroclinic: surfaces of constant density intersect surfaces of constant pressure



A baroclinic fluid induces a change in vorticity.

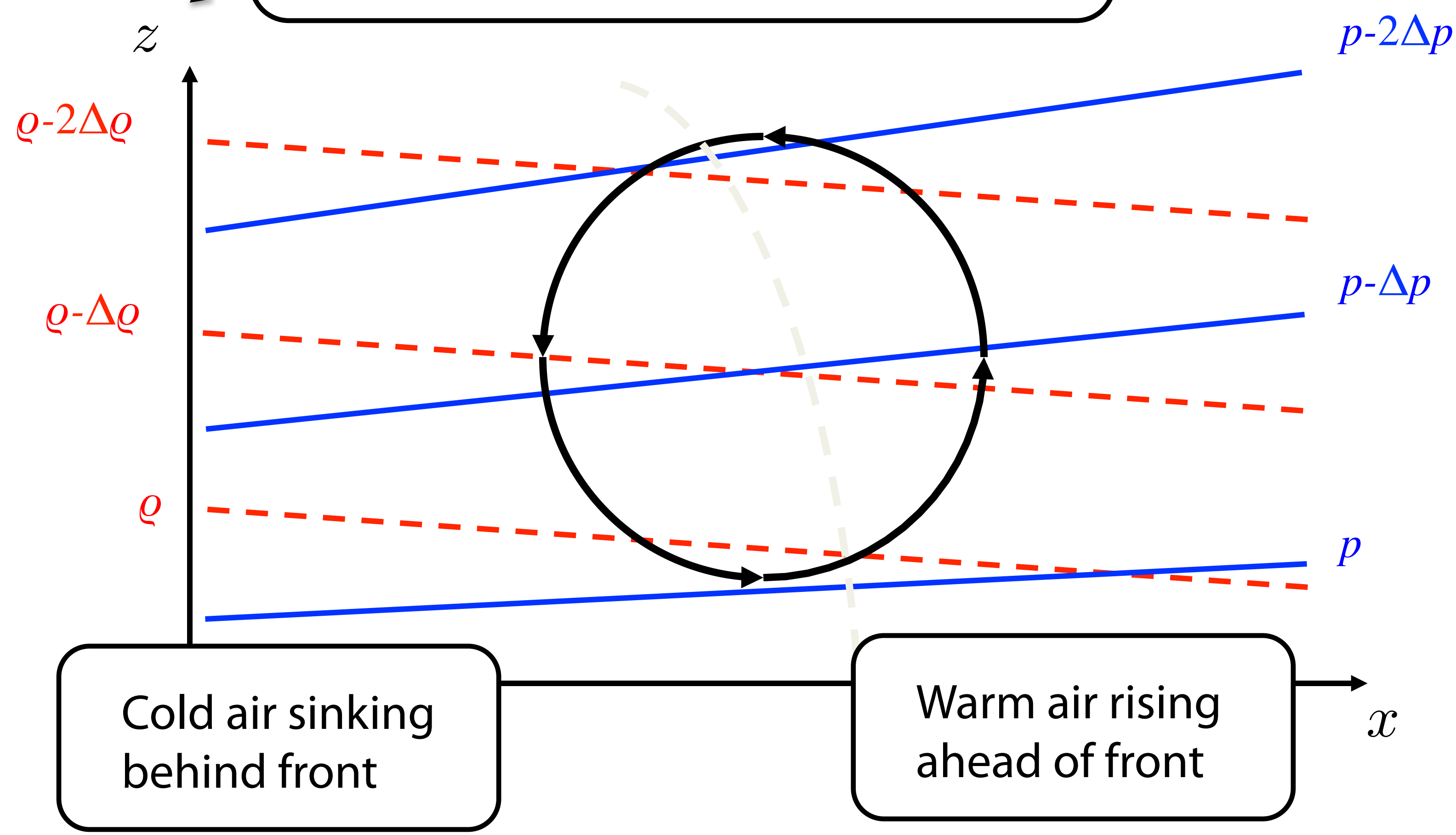
Solenoidal / Baroclinic Terms

In the real atmosphere, the solenoidal terms are a primary driver of the development of low-level low pressure systems in the middle latitudes.

This term is also important in understanding circulation around fronts.

Aside: Baroclinic Fronts

Note that this case has to do with the development of horizontal vorticity (perpendicular to x-z plane)



The Vorticity Equation

What are these terms?

$$\frac{D}{Dt}(\zeta + f) = -(\zeta + f)(\nabla_h \cdot \mathbf{u}) - \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z} \right) - (\nabla \alpha \times \nabla p) \cdot \mathbf{k}$$

Divergence term

Tilting or twisting term

Solenoidal term

Rate of change of the absolute vorticity following the motion

Advection of Vorticity

$$\frac{D}{Dt}(\zeta + f) = \frac{\partial}{\partial t}(\zeta + f) + u\frac{\partial}{\partial x}(\zeta + f) + v\frac{\partial}{\partial y}(\zeta + f) + w\frac{\partial}{\partial z}(\zeta + f)$$

$$\frac{\partial f}{\partial t} = 0 \qquad \frac{\partial f}{\partial x} = 0 \qquad \frac{\partial f}{\partial z} = 0$$

$$\frac{D}{Dt}(\zeta + f) = \frac{\partial \zeta}{\partial t} \boxed{+ \mathbf{u} \cdot \nabla \zeta} \boxed{+ v\frac{\partial f}{\partial y}}$$

Advection of relative vorticity

Advection of planetary vorticity

The Vorticity Equation

$$\frac{\partial \zeta}{\partial t} \boxed{+ \mathbf{u} \cdot \nabla \zeta} \boxed{+ (\zeta + f) \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)}$$

$$\boxed{+ \left(\frac{\partial w}{\partial x} \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} \frac{\partial u}{\partial z} \right)} \boxed{+ v \frac{\partial f}{\partial y}} = \boxed{- \frac{\partial}{\partial x} \left(\frac{1}{\rho} \right) \frac{\partial p}{\partial y} + \frac{\partial}{\partial y} \left(\frac{1}{\rho} \right) \frac{\partial p}{\partial x}}$$

Changes in absolute vorticity are caused by:

Advection	Divergence
Tilting	Baroclinicity

The Vorticity Equation

Scale Analysis

Changes in relative vorticity are caused by:

- Divergence
- Tilting
- Gradients in density
- Advection

Question: Which of these terms are the most important for large-scale flows?

Scale Analysis

Typical scales associated with large-scale mid-latitude storm systems:

$$U \approx 10 \text{ m s}^{-1}$$

$$W \approx 0.01 \text{ m s}^{-1}$$

$$L \approx 10^6 \text{ m}$$

$$H \approx 10^4 \text{ m}$$

$$L/U \approx 10^5 \text{ s}$$

$$\Delta P \approx 10 \text{ hPa} = 1000 \text{ Pa}$$

$$\rho \approx 1 \text{ kg m}^{-3}$$

$$\Delta\rho/\rho \approx 10^{-2}$$

$$f_0 \approx 10^{-4} \text{ s}^{-1}$$

$$\beta = \partial f/\partial y \approx 10^{-11} \text{ s}^{-1}$$

$a \approx 10^7 \text{ m}$ (Radius of Earth)

$g \approx 10 \text{ m s}^{-2}$ (Gravity)

$\nu \approx 10^{-5} \text{ m}^2 \text{ s}^{-1}$ (Kinematic Viscosity)

Scale Analysis

Relative Vorticity: $$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \approx \frac{U}{L} \approx 10^{-5}\ \mathrm{s}^{-1}$$

Planetary Vorticity: $$f_0 \approx 10^{-4}\ \mathrm{s}^{-1}$$

Definition: The **Rossby number** of a flow is a dimensionless quantity which represents the ratio of inertia to Coriolis force.

$$\frac{\zeta}{f_0} \approx \frac{U}{f_0 L} \equiv Ro$$

$$\Rightarrow \quad \frac{\zeta}{f_0} \approx 10^{-1}$$

In the mid-latitudes planetary vorticity is generally larger than relative vorticity.

Scale Analysis

Time rate of change and horizontal advection of relative vorticity:

$$\frac{\partial \zeta}{\partial t}, u\frac{\partial \zeta}{\partial x}, v\frac{\partial \zeta}{\partial y} \approx \frac{U^2}{L^2} \approx 10^{-10}\ \mathrm{s}^{-2}$$

Vertical advection of relative vorticity:

$$w\frac{\partial \zeta}{\partial z} \approx \frac{WU}{HL} \approx 10^{-11}\ \mathrm{s}^{-2}$$

Scale Analysis

Remaining Terms

Recall this is smaller than U/L

Divergence term: $$(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \approx f_0(\nabla_h \cdot \mathbf{u}) \approx 10^{-10}\ \mathrm{s}^{-2}$$

Tilting term: $$\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) \approx \frac{WU}{HL} \approx 10^{-11}\ \mathrm{s}^{-2}$$

Planetary vorticity advection: $$v\frac{\partial f}{\partial y} \approx U\beta \approx 10^{-10}\ \mathrm{s}^{-2}$$

Solenoidal term: $$\mathbf{k} \cdot (\nabla\alpha \times \nabla p) \approx \frac{\Delta\rho\Delta p}{\rho^2 L^2} \approx 10^{-11}\ \mathrm{s}^{-2}$$

The Vorticity Equation

“Large” terms (10^{-10} s^{-2})

$$\boxed{\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} + v\frac{\partial f}{\partial y} + (\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)}$$

$$= -w\frac{\partial \zeta}{\partial z} - \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) - \frac{\partial \alpha}{\partial x}\frac{\partial p}{\partial y} + \frac{\partial \alpha}{\partial y}\frac{\partial p}{\partial x}$$

“Small” terms (10^{-11} s^{-2})

The Vorticity Equation

Scale Analysis

Divergence term dominates along with **horizontal advection** and the **local time rate of change** of relative vorticity.

Tilting term important where there is a large shear and strong horizontal gradient in the vertical velocity (boundary layer, smaller scales).

Solenoidal term important where there are strong density (temperature) gradients that intersect lines of constant pressure (sea breeze, fronts).

The Vorticity Equation

Relative Vorticity, Planetary Vorticity and Divergence

$$\frac{f_0}{\zeta} \approx 10 \qquad \frac{\zeta}{\nabla_h \cdot \mathbf{u}} \approx 10 \qquad \frac{f_0}{\nabla_h \cdot \mathbf{u}} \approx 100$$

The rotation of the Earth is about 10 times larger than relative vorticity and 100 times larger than divergence.

The Vorticity Equation

Retaining leading order terms…

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} + v\frac{\partial f}{\partial y} + (\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

Definition: The horizontal material derivative is defined by the equation

$$\frac{D_h}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$$

$$\frac{D_h}{Dt}(\zeta + f) = -(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$