A Rotational View of the Atmoshere

Chapter 4

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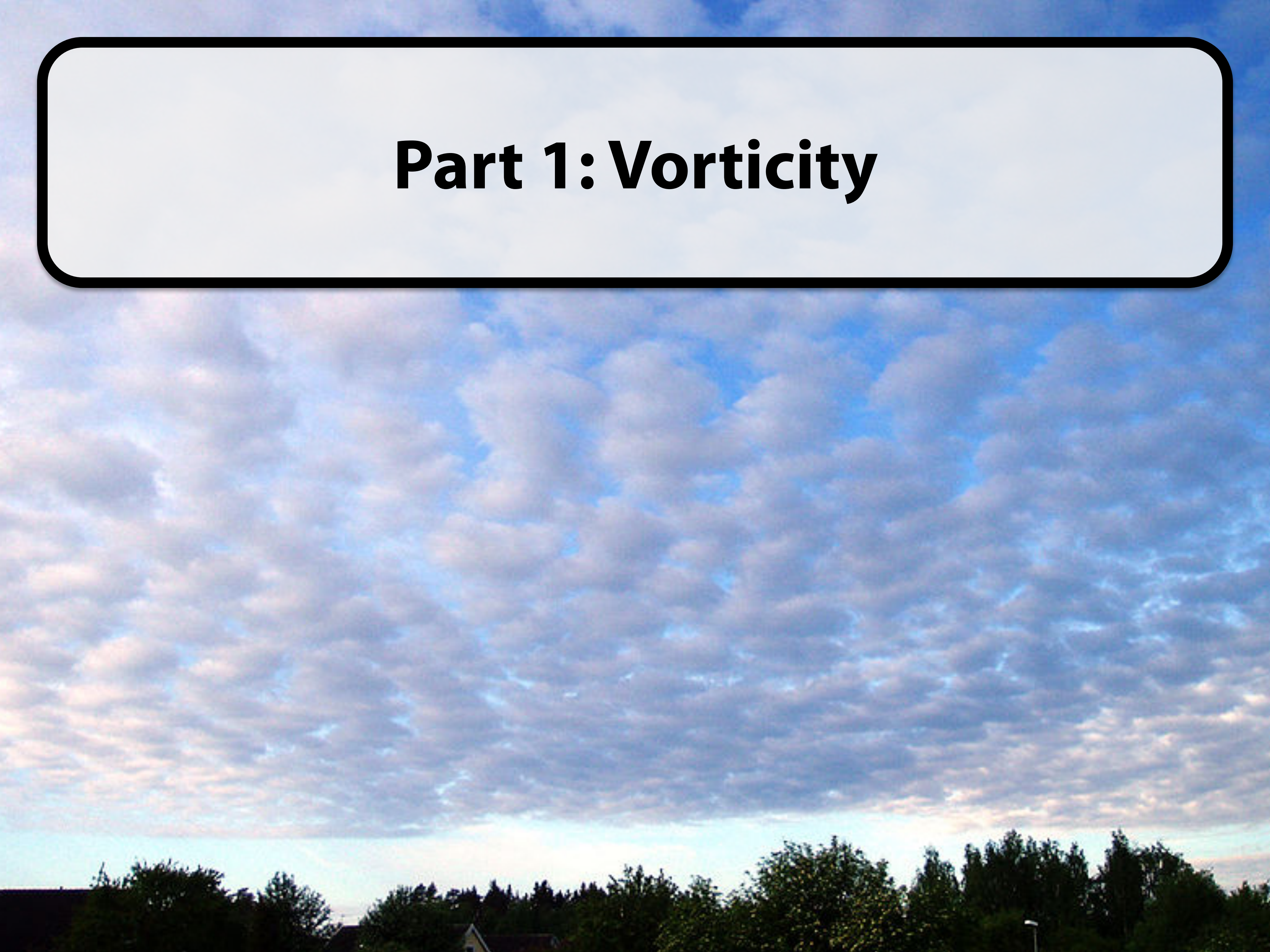
Part 1: Vorticity



Question: What are the goals of dynamic meteorology?

1. Understand the structure of atmospheric motions (diagnosis)

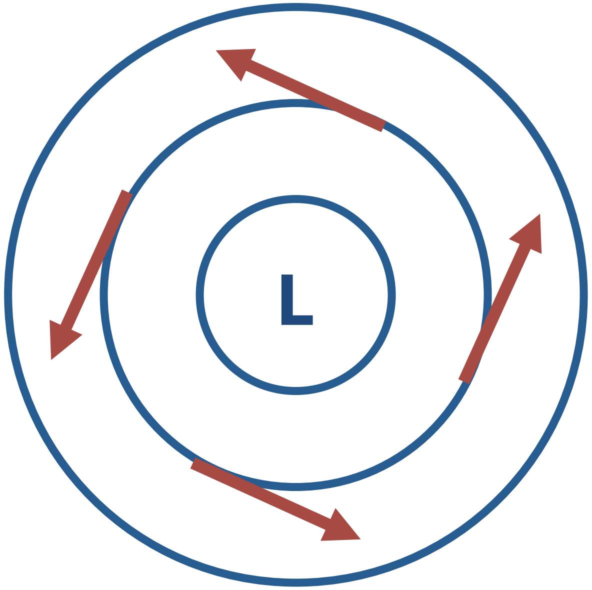
2. Predict future atmospheric motions (prognosis)

We will now focus on prognosis

Wind Around a System

At the simplest level, the atmosphere can be described using **vortex dynamics**.

Definition: A **vortex** is an area of closed, circular or near-circular fluid motion.

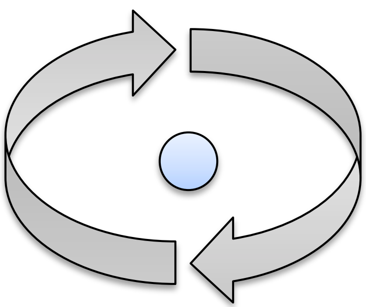


Definition: **Vorticity** is a measure of the local spinning motion of the flow. It consists of a vector which denotes the local axis of rotation and the local magnitude (or rotation rate).

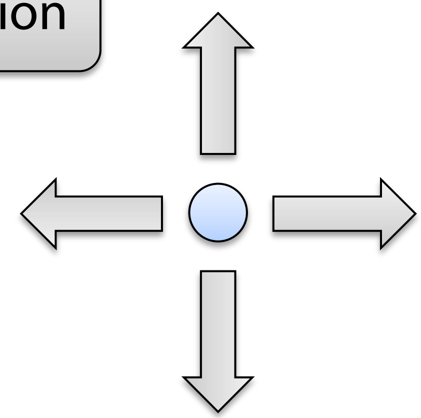
Rotation in the Flow

In accordance with the Helmholtz theorem, 2D fluid motion can be thought of as being composed of a **rotational component** and a **divergent component**.

Rotational Motion



Divergent Motion

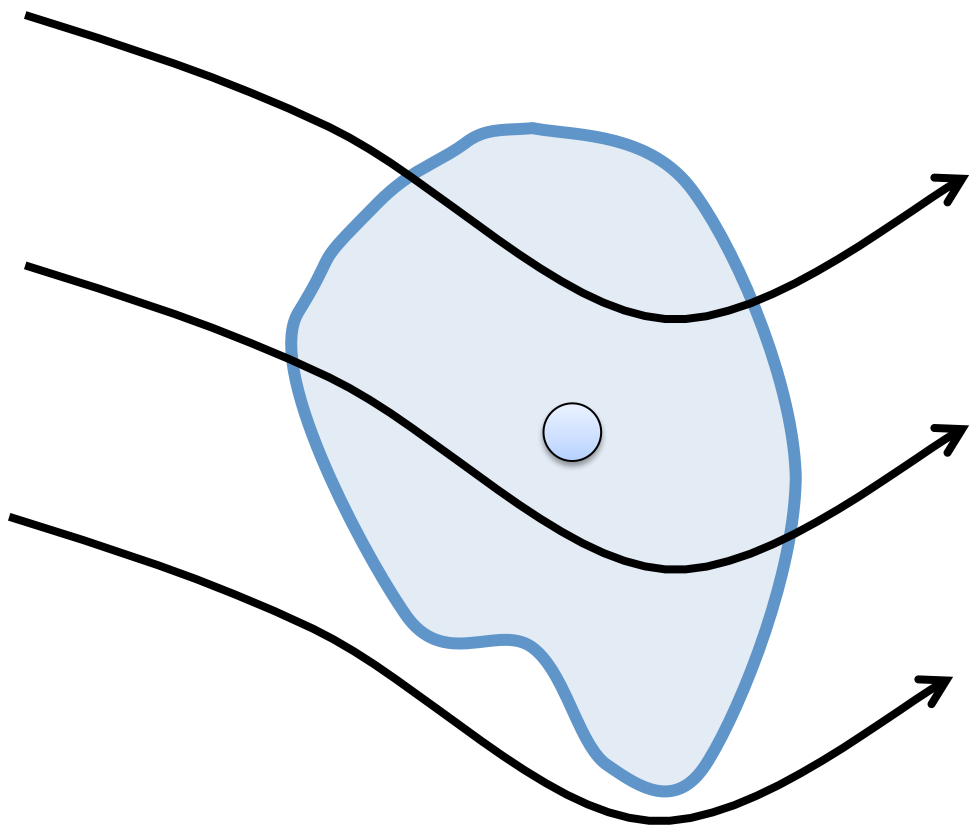


Rotation is important in the development of high and low pressure systems.

There is an interplay between rotation, divergence and vertical motion.

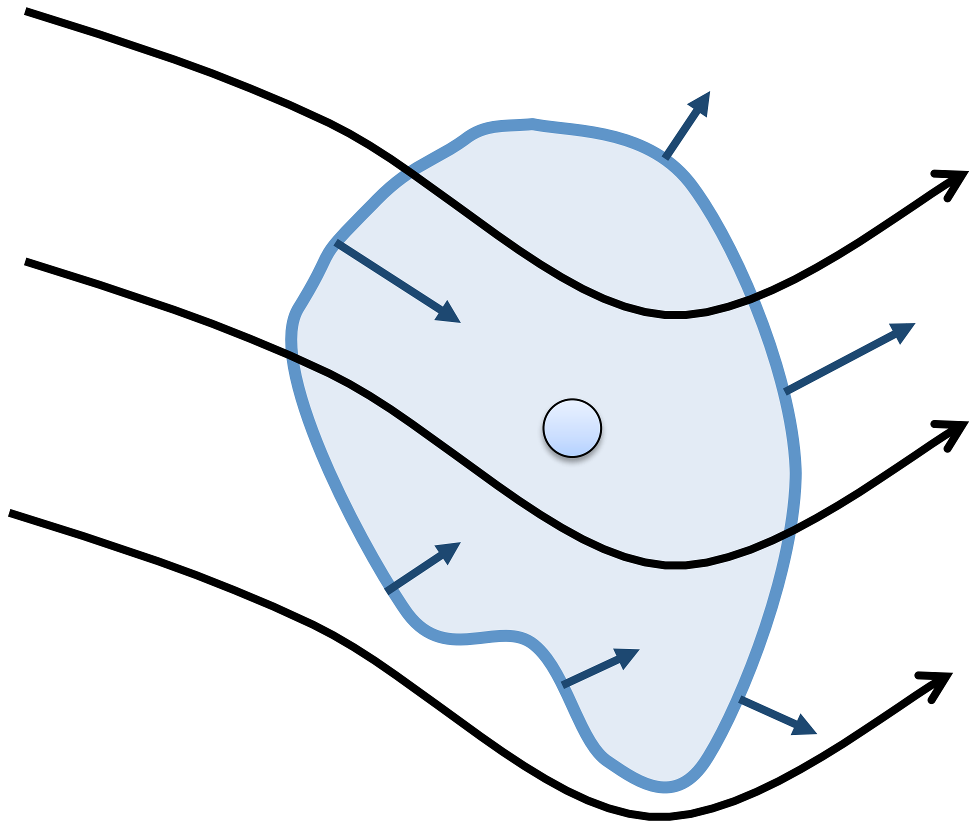
Types of Flow

Consider an arbitrary Eulerian region embedded in a flow.



Divergent Motion

Fluid motion associated with this region can be described in terms of **(a)** the motion of fluid in / out of this region (flux).



Gauss' Theorem

Flow into / out of a region must be associated with a particular **region**.

On the other hand, **divergence** is a closely connected concept which is described **at a point**.

These concepts are connected via Gauss' theorem:

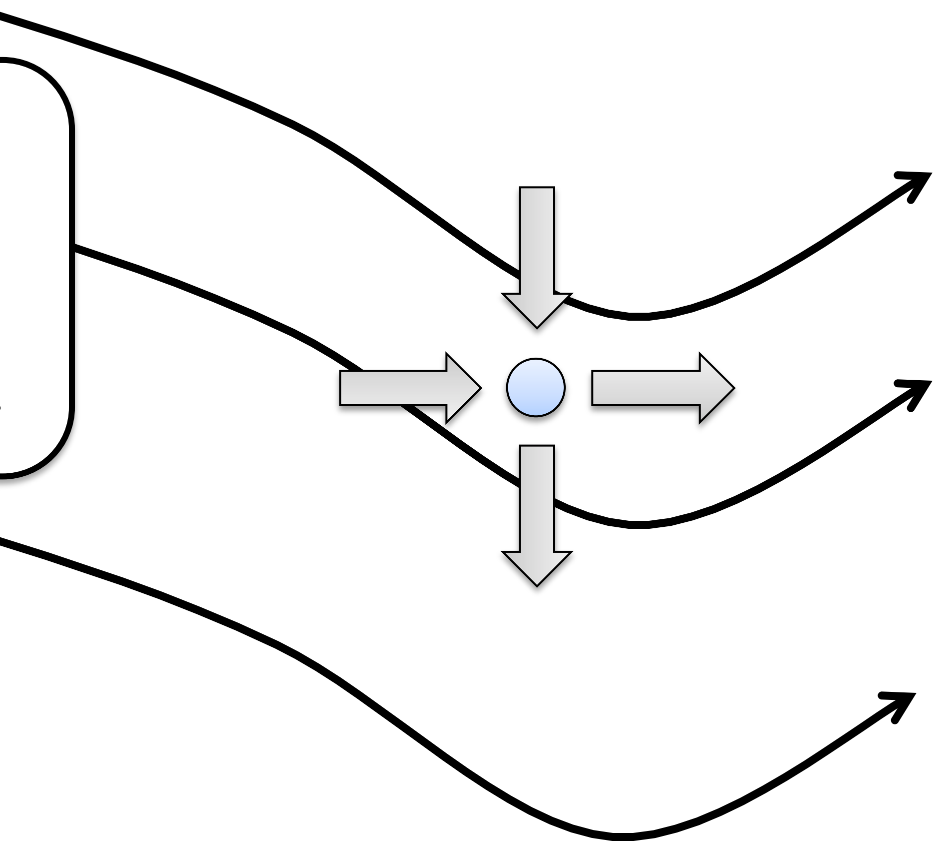
$$\int \mathbf{u} \cdot d\mathbf{n} = \int (\nabla \cdot \mathbf{u}) dV$$

Flow in / out of a region

Divergence of the flow field integrated over the region

Divergent Motion

The **divergent component** of the flow is used to describe flow into or away from the point.



Divergent Motion

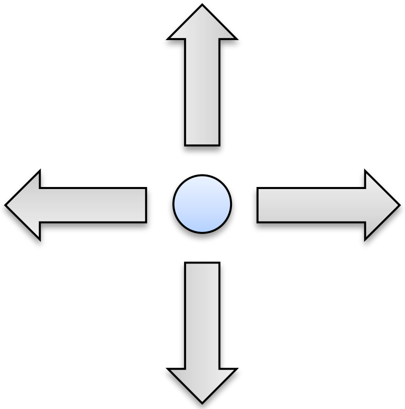
The **3D divergence operator** has the form

$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

The **2D divergence operator** (in the horizontal plane) has the form

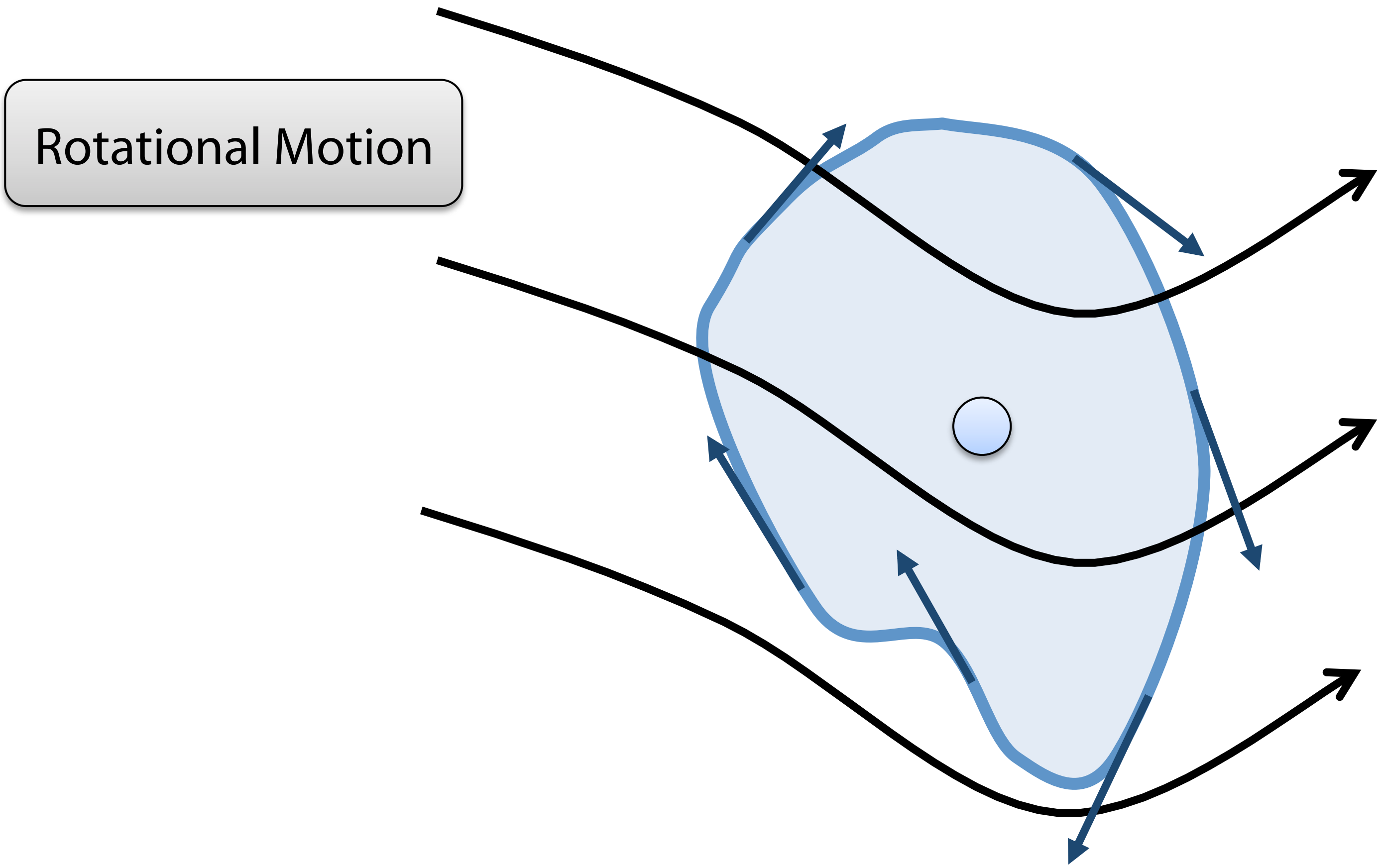
$$\nabla_h \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

Note the subscript



Rotational Motion

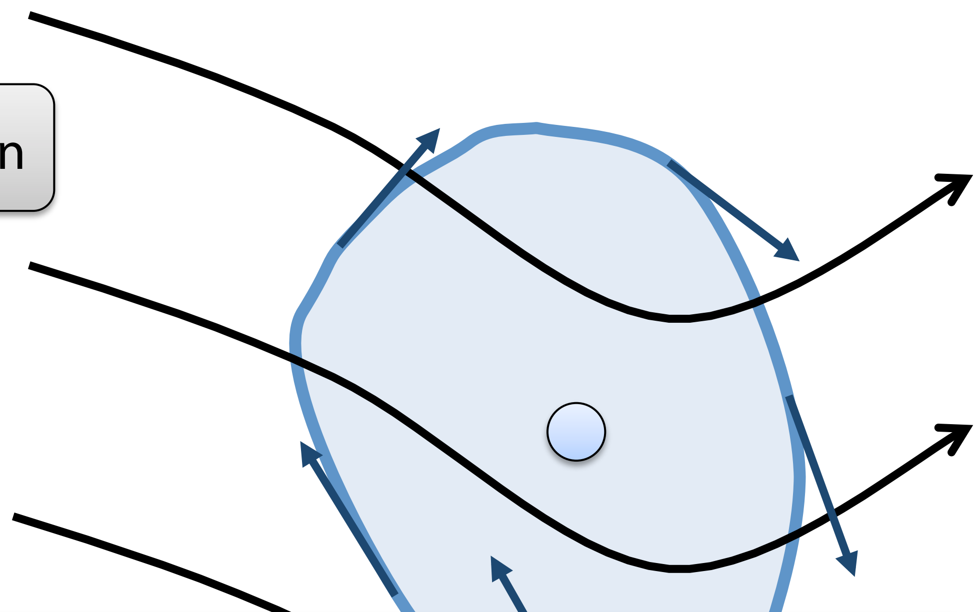
Fluid motion associated with this region can be described in terms of **(b)** the motion of fluid around this region.



Circulation

Fluid motion associated with this region can be described in terms of **(b)** the motion of fluid around this region.

Rotational Motion



Definition: The **circulation** of a flow about some curve is the integral of the tangental velocity around that curve.

$$C \equiv \oint \mathbf{u} \cdot d\mathbf{l}$$

Circulation

$$C \equiv \oint \mathbf{u} \cdot d\mathbf{l}$$

Circulation is a measure of rotational part of the flow within a region.

It is an analogue to angular momentum, and so induces a conservation law which describes its evolution. The "direction" of a circulation is defined by which direction we go around the circuit.

Cyclonic motion	Counterclockwise (in the Northern hemisphere)
Anticyclonic motion	Clockwise (in the Northern hemisphere)

Stokes' Theorem

Circulation is a concept which is valid along a **closed curve**.

On the other hand, **vorticity** is a closely connected concept which is described **at a point**.

These concepts are connected via Stokes' theorem:

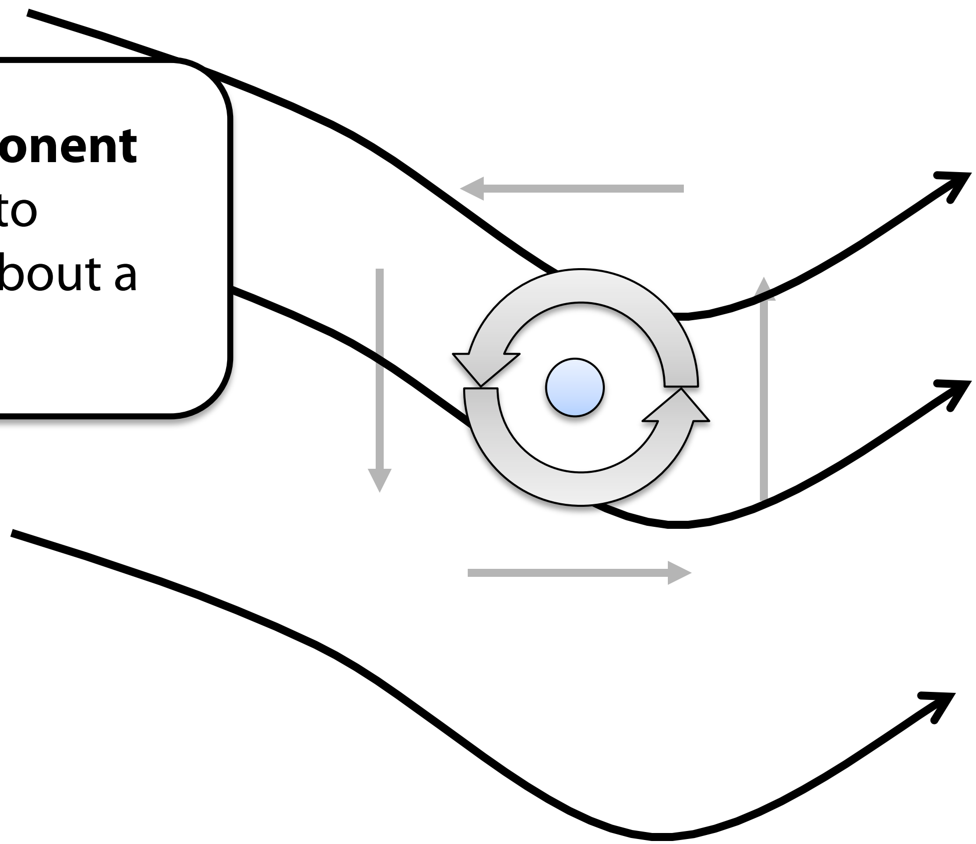
$$\oint \mathbf{u} \cdot d\mathbf{l} = \int (\nabla \times \mathbf{u}) \cdot d\mathbf{S}$$

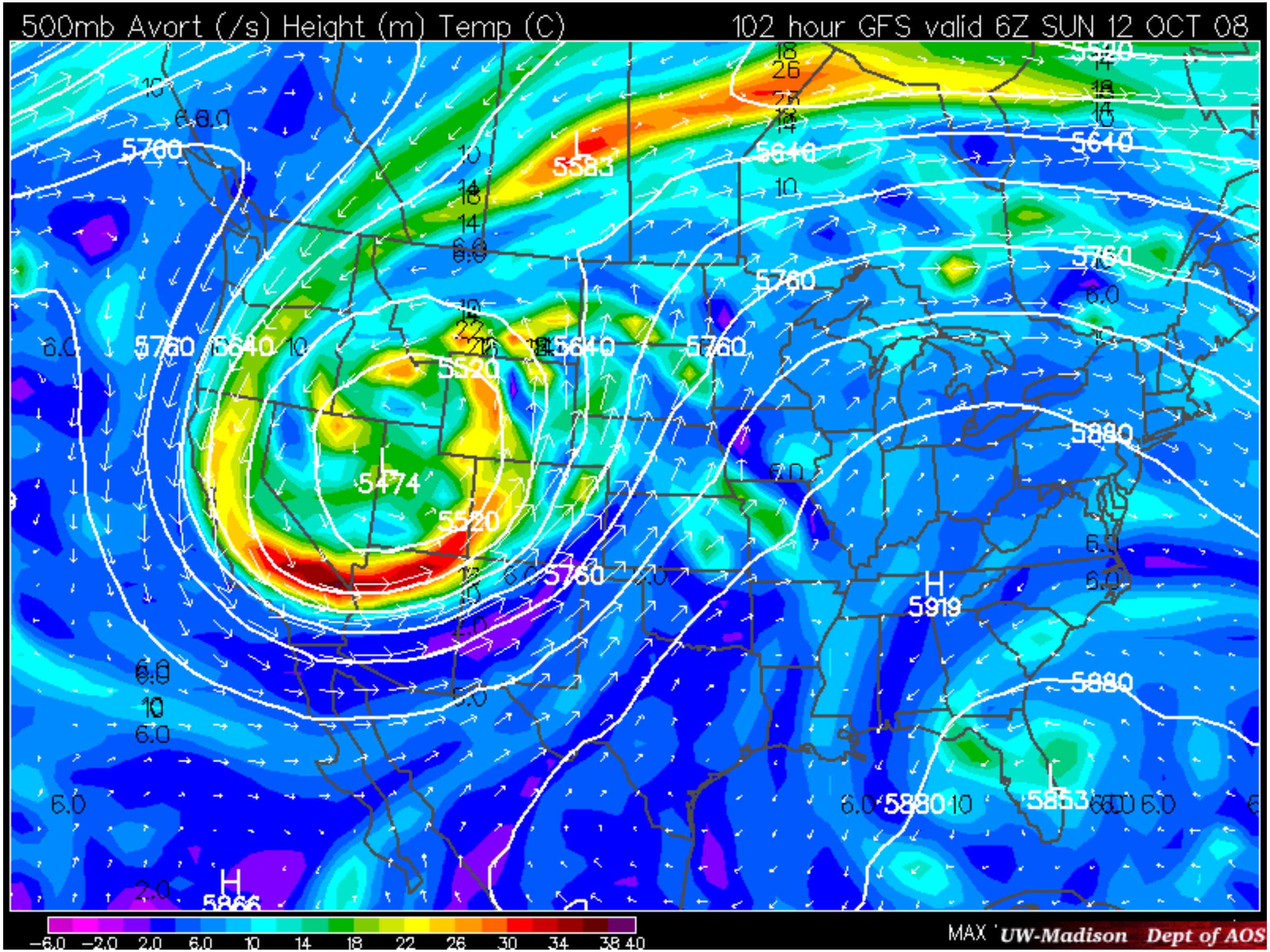
Circulation about a curve

Vorticity field over the surface bounded by the curve

Rotational Motion

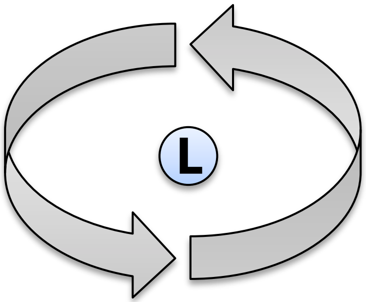
The **vortical component** of the flow is used to describe rotation about a point.




500mb Avort (/s) Height (m) Temp (C)
102 hour GFS valid 6Z SUN 12 OCT 08
UW-Madison Dept of AOS

Vorticity Concepts

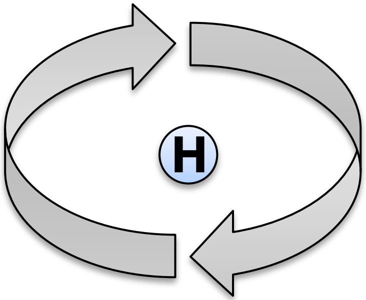
In accordance with geostrophic balance, **positive vorticity** is associated with **cyclonic rotation** in the northern hemisphere (low pressure systems)



Predictions to changes in vorticity are then equivalent to predictions in changes in pressure.

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

The first computer forecasts only predicted the changes in vorticity and still did a decent job…



Analogously, **negative vorticity** is associated with **anti-cyclonic rotation** in the northern hemisphere (high pressure systems)

Vorticity

The **3D vorticity** is associated with the curl of the vector field:

$$\nabla \times \mathbf{u} = \mathbf{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \mathbf{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \mathbf{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

These components represent vorticity due to **overturning motions** and **vertical wind shear**.

This component represents vorticity confined to a horizontal surface. This type of motion is the most important for understanding the dynamic evolution of large-scale weather systems.

Vorticity

For diagnostic purposes, the vertical component of vorticity is the most important:

$$\mathbf{k} \cdot (\nabla \times \mathbf{u})$$

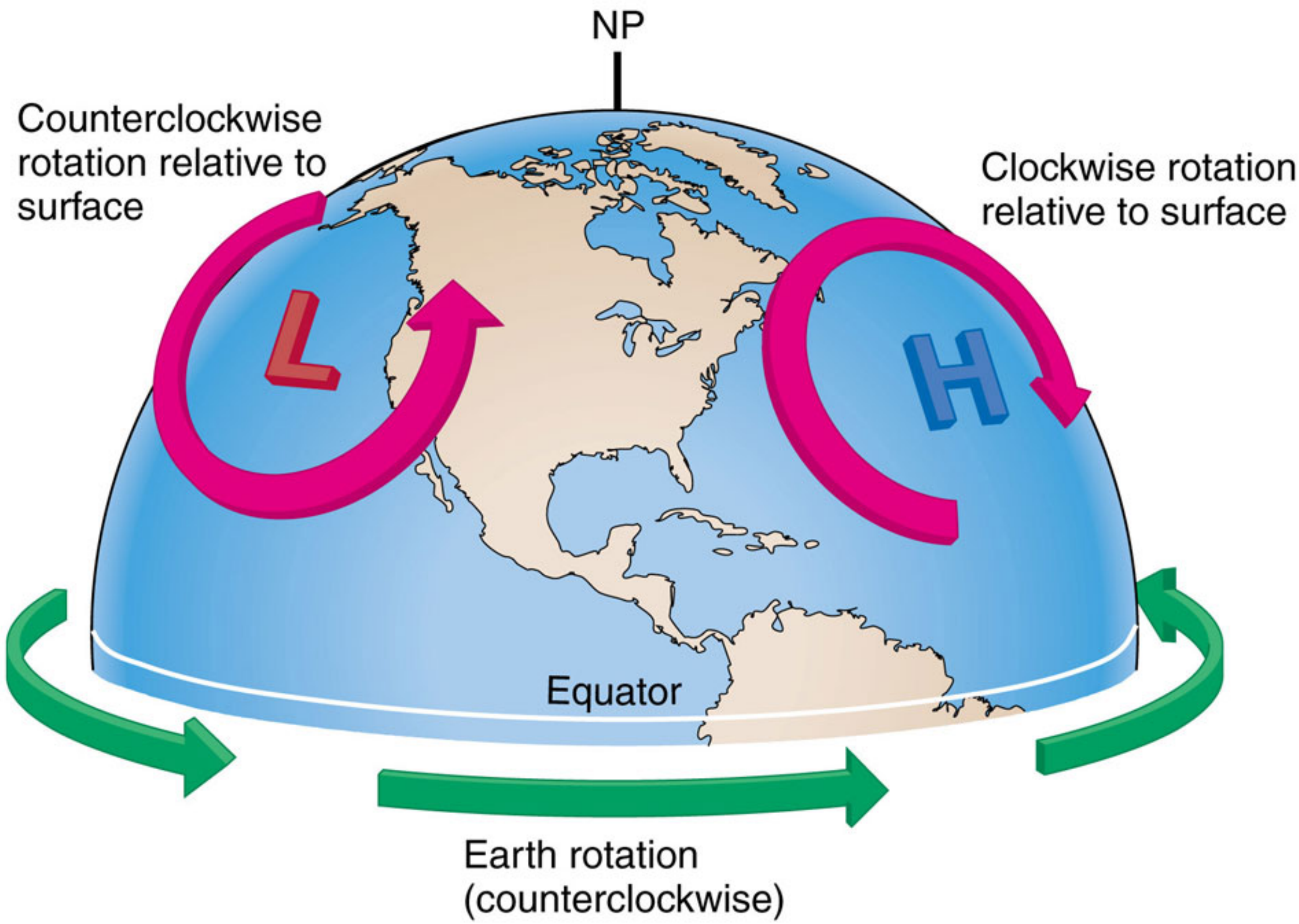
Hence, the **2D vorticity** is associated with the quantity:

$$\boxed{\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}}$$

This is the **relative vorticity**, since it is computed locally and does not take into account the rotation of the coordinate system.

Vorticity

Consider only the horizontal component of the vorticity:



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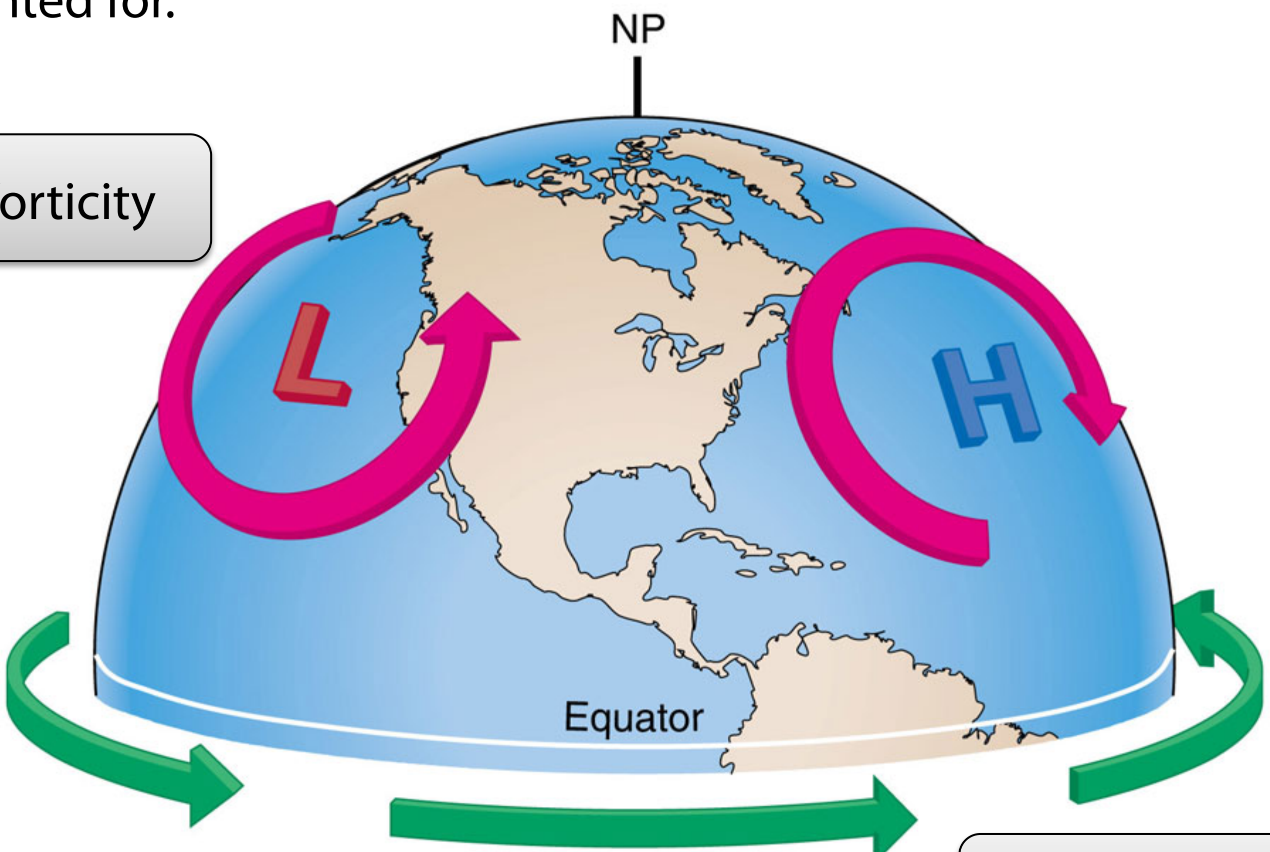
Since $\mathbf{u}$ is the velocity in the inertial coordinate system, both rotation of the planet and relative rotation need to be accounted for:

Definition: The component of vorticity due to relative motions is called **relative vorticity**.

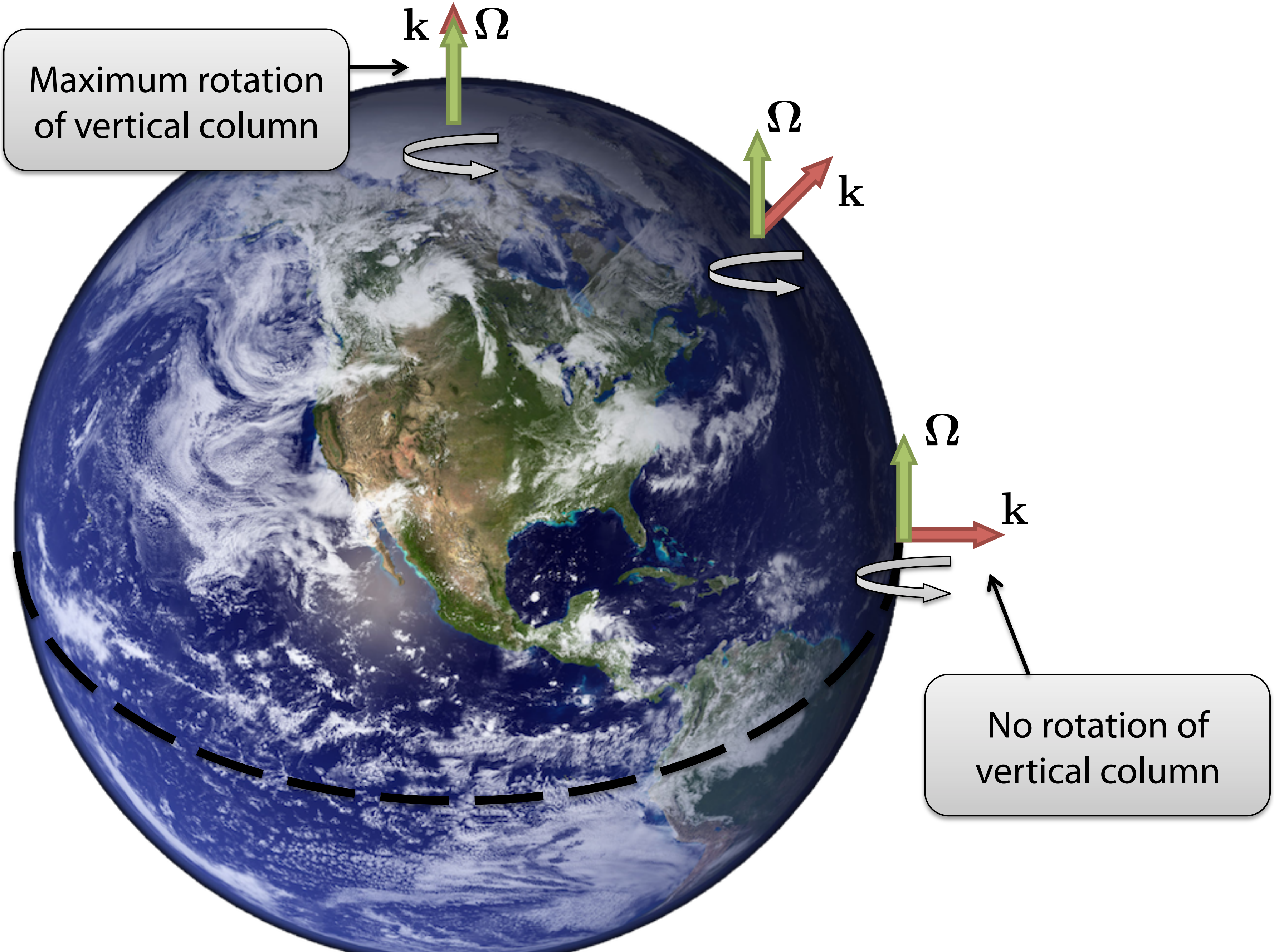
Vorticity

Since angular momentum is only conserved in an inertial frame, rotation of the coordinate system must be accounted for.



Relative Vorticity

Planetary Vorticity

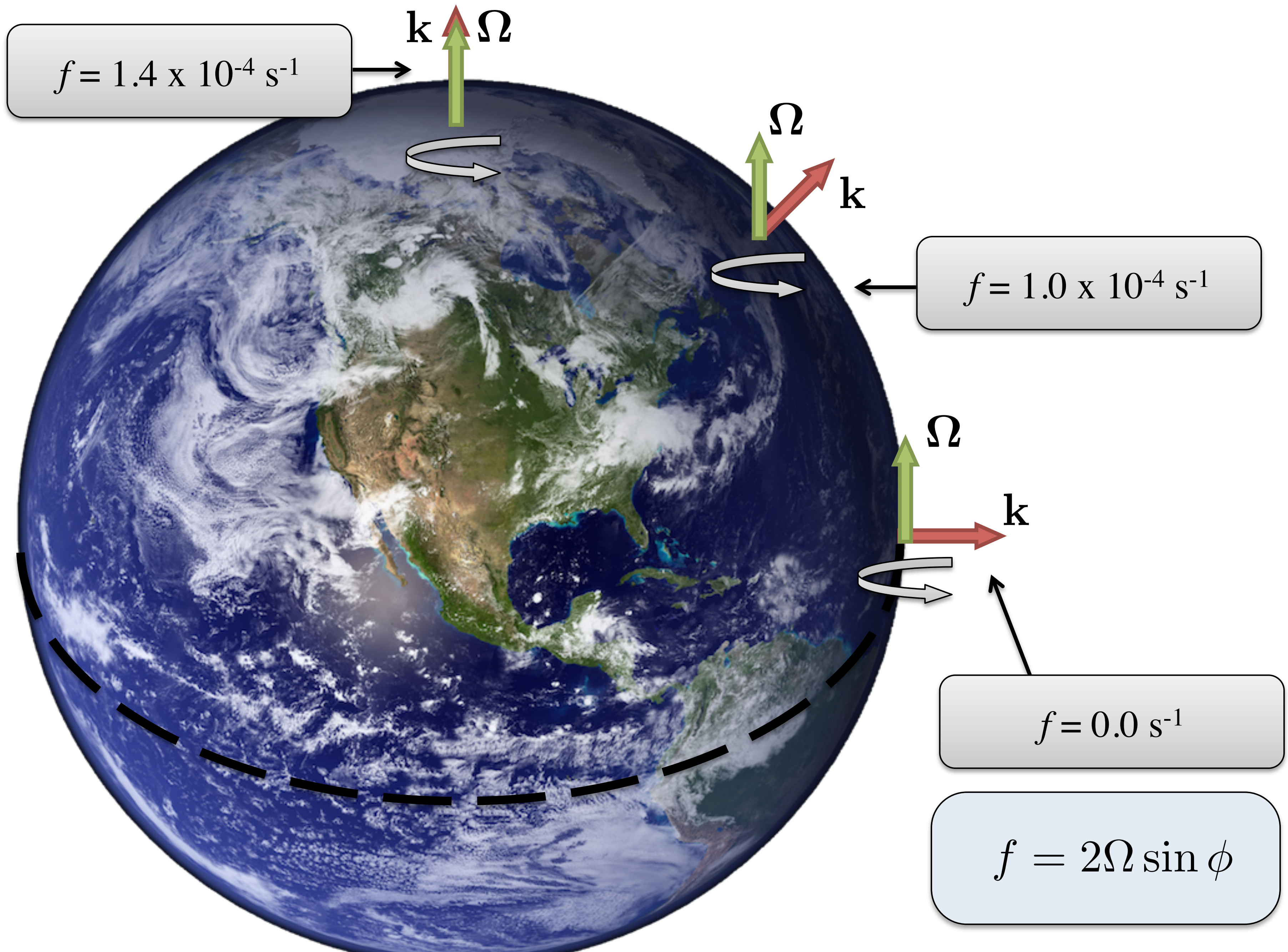

Maximum rotation of vertical column
k
Ω
Ω
k
Ω
k
No rotation of vertical column

Planetary Vorticity

Definition: The component of vorticity due to the rotation of the Earth is called **planetary vorticity**.

$$f = \mathbf{k} \cdot (\nabla \times \mathbf{u}_{earth}) = 2\Omega \sin \phi$$

Planetary vorticity is the contribution to angular momentum due to the rotation of the planetary surface.


$f = 1.4 \times 10^{-4}\ \mathrm{s}^{-1}$
k
Ω
Ω
k
$f = 1.0 \times 10^{-4}\ \mathrm{s}^{-1}$
Ω
k
$f = 0.0\ \mathrm{s}^{-1}$
$f = 2\Omega \sin \phi$

Absolute Vorticity

Definition: The **absolute (or total) vorticity (η)** is the sum of the relative vorticity and planetary vorticity.

Absolute Vorticity = Relative Vorticity + Planetary Vorticity

$$\eta \equiv \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) + f$$

This concept arises immediately from $\mathbf{u}_a = \mathbf{u}_{earth} + \mathbf{u}$ and linearity of the curl:

$$\mathbf{k} \cdot (\nabla \times \mathbf{u}_a) = \mathbf{k} \cdot (\nabla \times \mathbf{u}_{earth}) + \mathbf{k} \cdot (\nabla \times \mathbf{u})$$

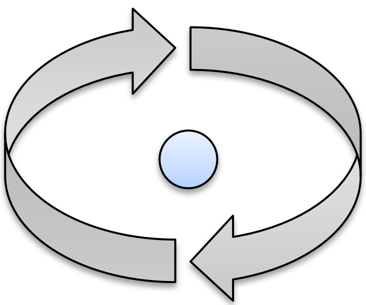
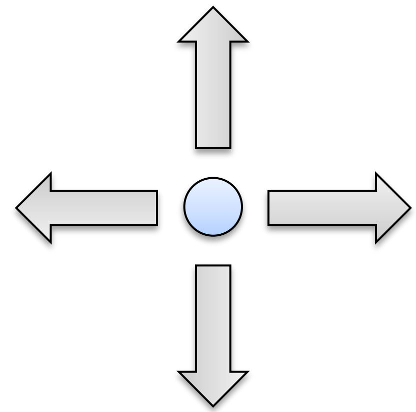
Divergence

What about the contribution of the Earth's rotation to the divergence?

$$\begin{aligned}
\nabla \cdot \mathbf{u}_{earth} &= \nabla \cdot (2\Omega \sin\phi \mathbf{i}) \\
&= \frac{\partial}{\partial x}(2\Omega \sin\phi) \\
&= \frac{1}{r\cos\phi}\frac{\partial}{\partial\lambda}(2\Omega\sin\phi) \\
&= 0
\end{aligned}$$

The rotation of the coordinate system does not contribute to the divergence of the flow.

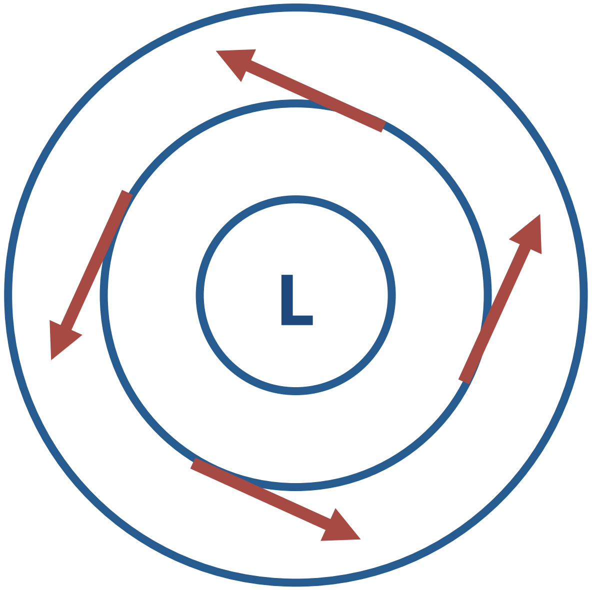
Rotational vs. Divergent Flow

Rotational Motion		Divergent Motion
		
$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$	**Associated 2D Diagnostic Variable**	$\delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$
~ 10^{-5} s^{-1}	**Typical Scale**	**~ 10^{-6} s^{-1}**

Rotation is an order of magnitude more important than divergence!

Vorticity Concepts

At the simplest level, the atmosphere can be described using **vortex dynamics**. Vortices are dominated by rotation (curvature).



Question: Do all curved flows have vorticity? Are all flows with vorticity curved?

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

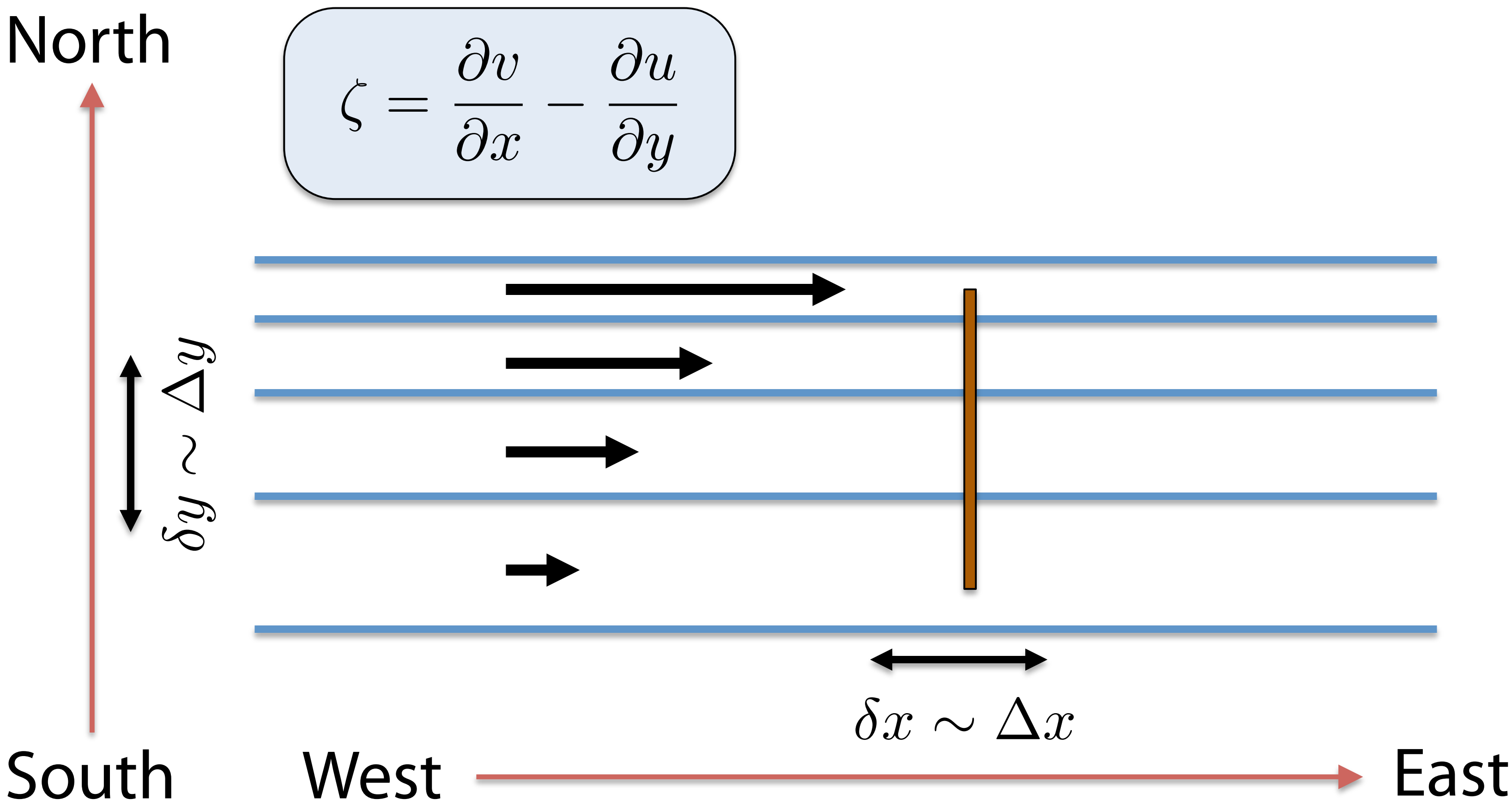
Observe: Vorticity is associated with pointwise spinning motion.

Answer: No. Not all curved flows have vorticity. Further, not all flows with vorticity are curved.

Vorticity Concepts

Vorticity is associated with pointwise spinning motion. So what happens to this stick?

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$



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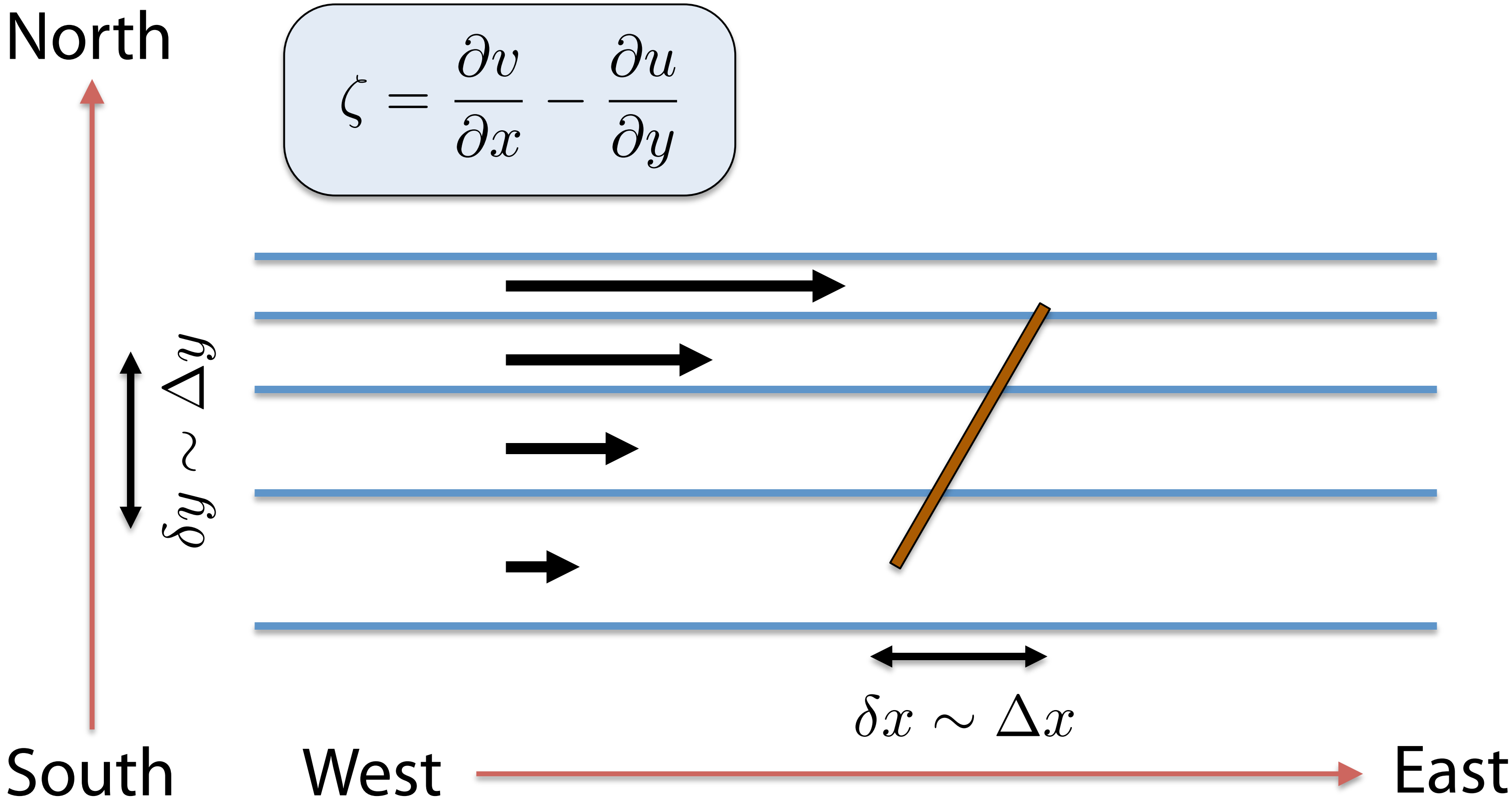

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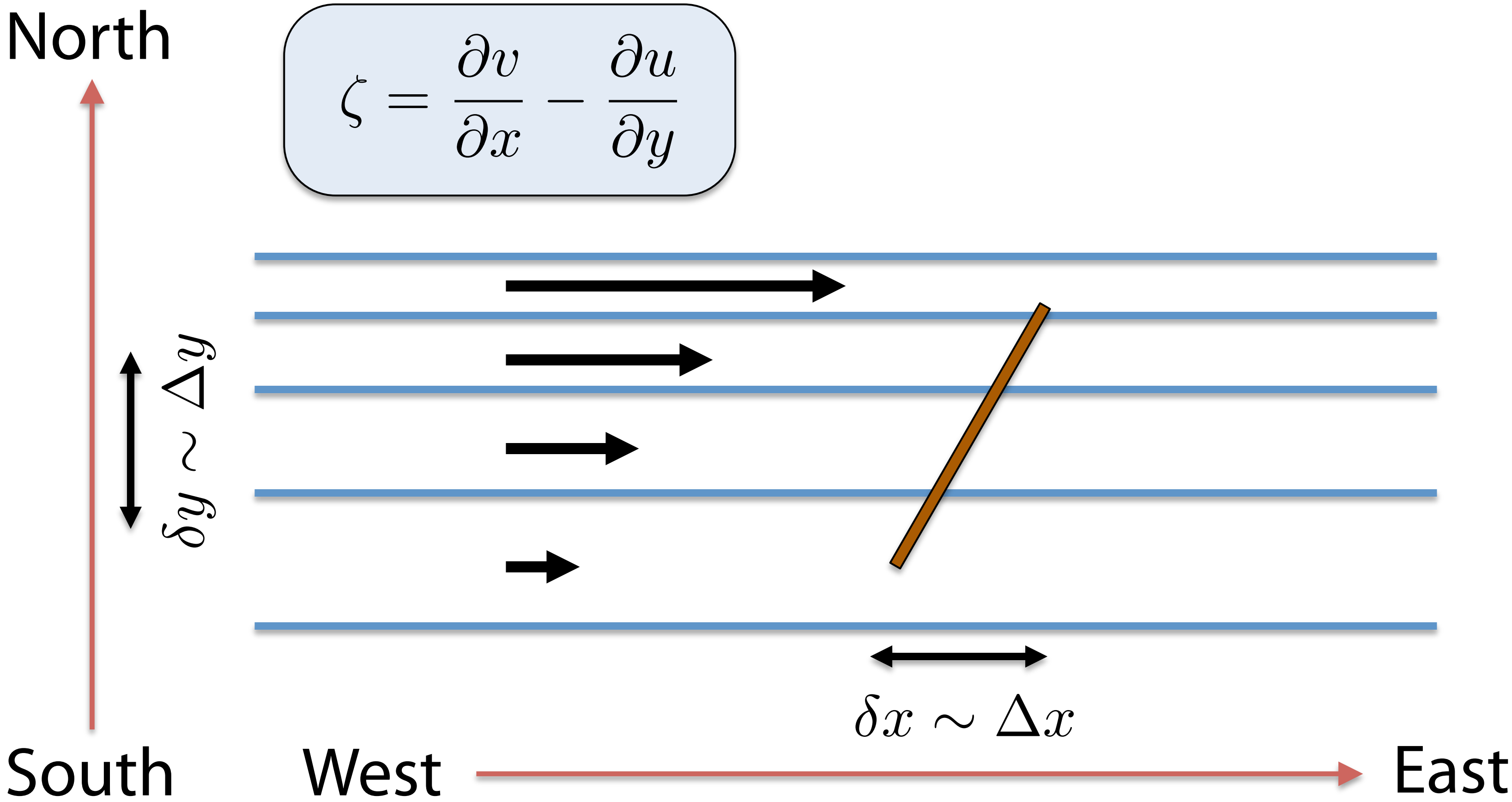

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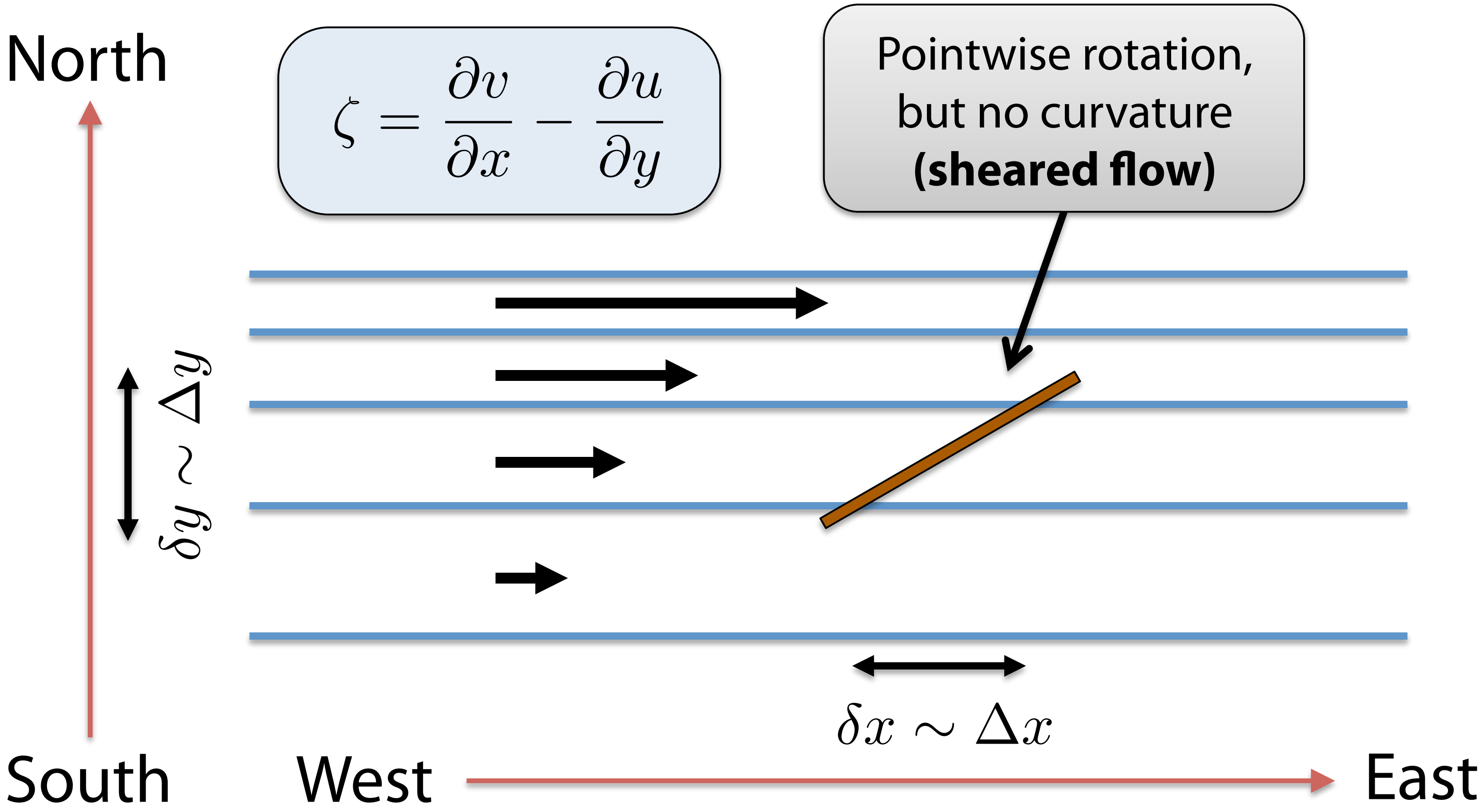

Vorticity Concepts

Relative velocity: $$\mathbf{u} = (u, v, w)$$

3D vorticity vector: $$\nabla \times \mathbf{u} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$

Relative vorticity: $$\zeta = \mathbf{k} \cdot (\nabla \times \mathbf{u}) = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$

Absolute vorticity: $$\eta = \mathbf{k} \cdot (\nabla \times \mathbf{u}_a) = \zeta + f$$

Planetary vorticity: $$f = \mathbf{k} \cdot (\nabla \times \mathbf{u}_{earth}) = 2\Omega \sin \phi$$

(local vertical component of the vorticity of the Earth)