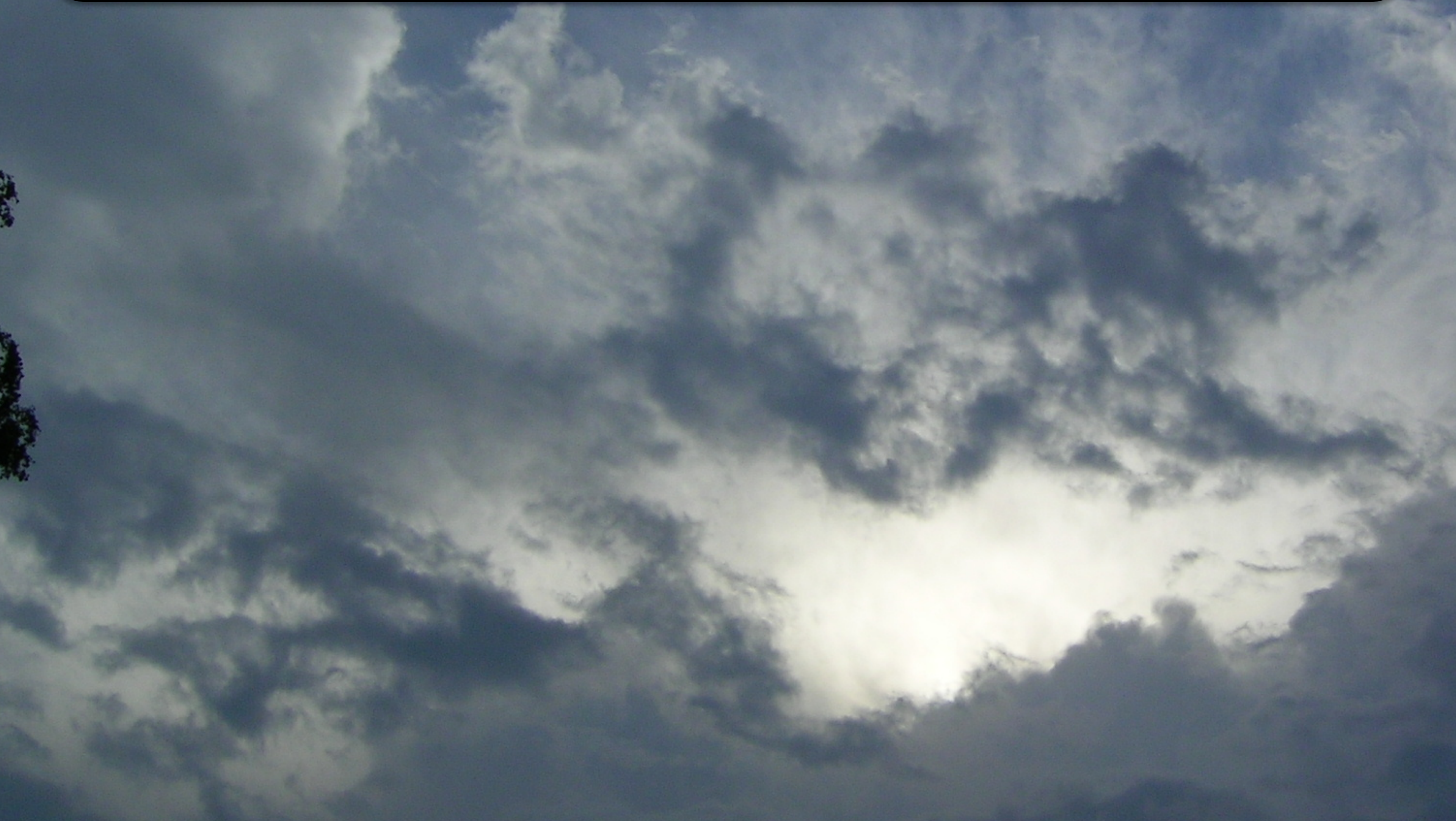
Applications of the Basic Equations

Chapter 3

Paul A. Ullrich

paullrich@ucdavis.edu

Part 4: Vertical Motion in Pressure Coordinates



Dynamical Equations

Pressure Coordinates

Momentum Equation

$$\frac{D\mathbf{u}_h}{Dt} + f\mathbf{k} \times \mathbf{u}_h = -\nabla_p \Phi$$

Hydrostatic Relation

$$\frac{\partial \Phi}{\partial p} = -\frac{R_d T}{p}$$

Continuity Equation

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)_p + \frac{\partial \omega}{\partial p} = 0$$

Thermodynamic Equation

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} - S_p \omega = \frac{J}{c_p}$$

Ideal Gas Law

$$p = \rho R_d T$$

Material Derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial p}$$

Vertical Motion

Connection between w and ω

By definition,

$$\omega \equiv \frac{Dp}{Dt} = \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z}$$

Hydrostatic Balance

$$\boxed{\frac{\partial p}{\partial z} = -\rho g}$$

$$\omega = \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} - wg\rho$$

Geostrophic Decomposition

$$\boxed{\mathbf{u} = \mathbf{u}_g + \mathbf{u}_a}$$

$$\omega = \frac{\partial p}{\partial t} + (u_g + u_a)\frac{\partial p}{\partial x} + (v_g + v_a)\frac{\partial p}{\partial y} - wg\rho$$

Vertical Motion

Connection between w and ω

$$\omega = \frac{\partial p}{\partial t} + (u_g + u_a)\frac{\partial p}{\partial x} + (v_g + v_a)\frac{\partial p}{\partial y} - wg\rho$$

Geostrophic Balance

$$u_g = -\frac{1}{f\rho}\frac{\partial p}{\partial y}$$

$$v_g = +\frac{1}{f\rho}\frac{\partial p}{\partial x}$$

$$\omega = \frac{\partial p}{\partial t} + \left(u_a\frac{\partial p}{\partial x} + v_a\frac{\partial p}{\partial y}\right) - wg\rho$$

Question: What are the scales associated with these terms?

Scale Analysis

Typical scales associated with large-scale mid-latitude storm systems:

$$U \approx 10 \text{ m s}^{-1}$$

$$W \approx 0.01 \text{ m s}^{-1}$$

$$L \approx 10^6 \text{ m}$$

$$H \approx 10^4 \text{ m}$$

$$L/U \approx 10^5 \text{ s}$$

$$\Delta P \approx 10 \text{ hPa} = 1000 \text{ Pa}$$

$$\rho \approx 1 \text{ kg m}^{-3}$$

$$\Delta\rho/\rho \approx 10^{-2}$$

$$f_0 \approx 10^{-4} \text{ s}^{-1}$$

$a \approx 10^7 \text{ m}$ (Radius of Earth)

$g \approx 10 \text{ m s}^{-2}$ (Gravity)

$\nu \approx 10^{-5} \text{ m}^2 \text{ s}^{-1}$ (Kinematic Viscosity)

Vertical Motion

Connection between *w* and *ω*

$$\omega = \frac{\partial p}{\partial t} + \left(u_a \frac{\partial p}{\partial x} + v_a \frac{\partial p}{\partial y} \right) - wg\rho$$

Local change in pressure: $$\frac{\partial p}{\partial t} \approx \frac{U \Delta P}{L} \approx 10^{-2} \text{ Pa s}^{-1}$$

Pressure advection by ageostrophic wind: $$\mathbf{u}_a \cdot \nabla_h p \approx 0.1 \times \frac{U \Delta P}{L} \approx 10^{-3} \text{ Pa s}^{-1}$$

Ageostrophic velocity is "small"

Vertical velocity term: $$wg\rho \approx Wg\rho \approx 10^{-1} \text{ Pa s}^{-1}$$

Vertical Motion

Connection between w and ω

From scale analysis, to a close approximation the vertical pressure velocity is given by this relationship.

$$\omega \approx -wg\rho$$

Vertical Motion

Question: How do we diagnose the vertical pressure velocity?

Observe there are only two equations that include ω:

Continuity Equation

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)_p + \frac{\partial \omega}{\partial p} = 0$$

Thermodynamic Equation

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} - S_p\omega = \frac{J}{c_p}$$

Each of these equations leads to one diagnostic equation for ω.

Vertical Pressure Velocity

Kinematic Method

Continuity Equation

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)_p + \frac{\partial \omega}{\partial p} = 0$$

Geostrophic Decomposition

$$\nabla_p \cdot (\mathbf{u}_g + \mathbf{u}_a) + \frac{\partial \omega}{\partial p} = 0$$

Geostrophic Balance

$$\frac{\partial}{\partial x}\left(-\frac{1}{f}\frac{\partial \Phi}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{1}{f}\frac{\partial \Phi}{\partial x}\right) + \nabla_p \cdot \mathbf{u}_a + \frac{\partial \omega}{\partial p} = 0$$

Vertical Pressure Velocity

Kinematic Method

$$\cancel{\frac{\partial}{\partial x}\left(-\frac{1}{f}\frac{\partial \Phi}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{1}{f}\frac{\partial \Phi}{\partial x}\right)} + \nabla_p \cdot \mathbf{u}_a + \frac{\partial \omega}{\partial p} = 0$$

Assume f is approximately constant

$$\frac{\partial \omega}{\partial p} \approx -\nabla_p \cdot \mathbf{u}_a$$

As observed for the case of height coordinates, vertical pressure velocity is connected to the divergence of the ageostrophic wind in pressure surfaces.

Vertical Pressure Velocity

Kinematic Method

$$\frac{\partial \omega}{\partial p} \approx -\nabla_p \cdot \mathbf{u}_a$$

Integrate over a pressure interval (p_1, p_2)

$$\omega(p_1) \approx \omega(p_2) - (p_1 - p_2)\left(\frac{\partial \langle u \rangle}{\partial x} + \frac{\partial \langle v \rangle}{\partial y}\right)$$

Using the Kinematic method, vertical pressure velocity is diagnosed from the mean layer horizontal divergence.

Vertical Pressure Velocity

Kinematic Method

$$\omega(p_1) \approx \omega(p_2) - (p_1 - p_2)\left(\frac{\partial\langle u\rangle}{\partial x} + \frac{\partial\langle v\rangle}{\partial y}\right)$$

However, if the flow is very close to geostrophic balance, then the divergence is small and calculating the mean layer divergence requires an accurate representation of horizontal velocities.

Therefore: Small errors in evaluating the winds <u> and <v> lead to large errors in ω. Consequently, the Kinematic method tends to be inaccurate.

Vertical Pressure Velocity

Adiabatic Method

Thermodynamic Equation

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} - S_p\omega = \frac{J}{c_p}$$

Assume diabatic heating is small

$$\omega = S_p^{-1}\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right)$$

Stability parameter

Horizontal advection of temperature

Vertical Pressure Velocity

Adiabatic Method

If local time tendency is negligible (steady state)

$$\frac{\partial T}{\partial t} \approx 0 \quad \Rightarrow \quad \omega = -\left[\frac{-\mathbf{u}_h \cdot \nabla T}{S_p}\right]$$

Horizontal advection of temperature

If temperature time tendency is steady, flow is adiabatic and the atmosphere is stable:

Warm air advection $\Rightarrow$ **Ascending Air**

$$\omega < 0, w \approx -\omega/\rho g > 0$$

Cold air advection $\Rightarrow$ **Descending Air**

$$\omega > 0, w \approx -\omega/\rho g < 0$$

Vertical Pressure Velocity

Adiabatic Method

The adiabatic method is based on temperature advection, which is dominated by the geostrophic wind (large).

Hence, this method is a reasonable way to estimate local vertical velocity when advection is strong.