Applications of the Basic Equations

Chapter 3

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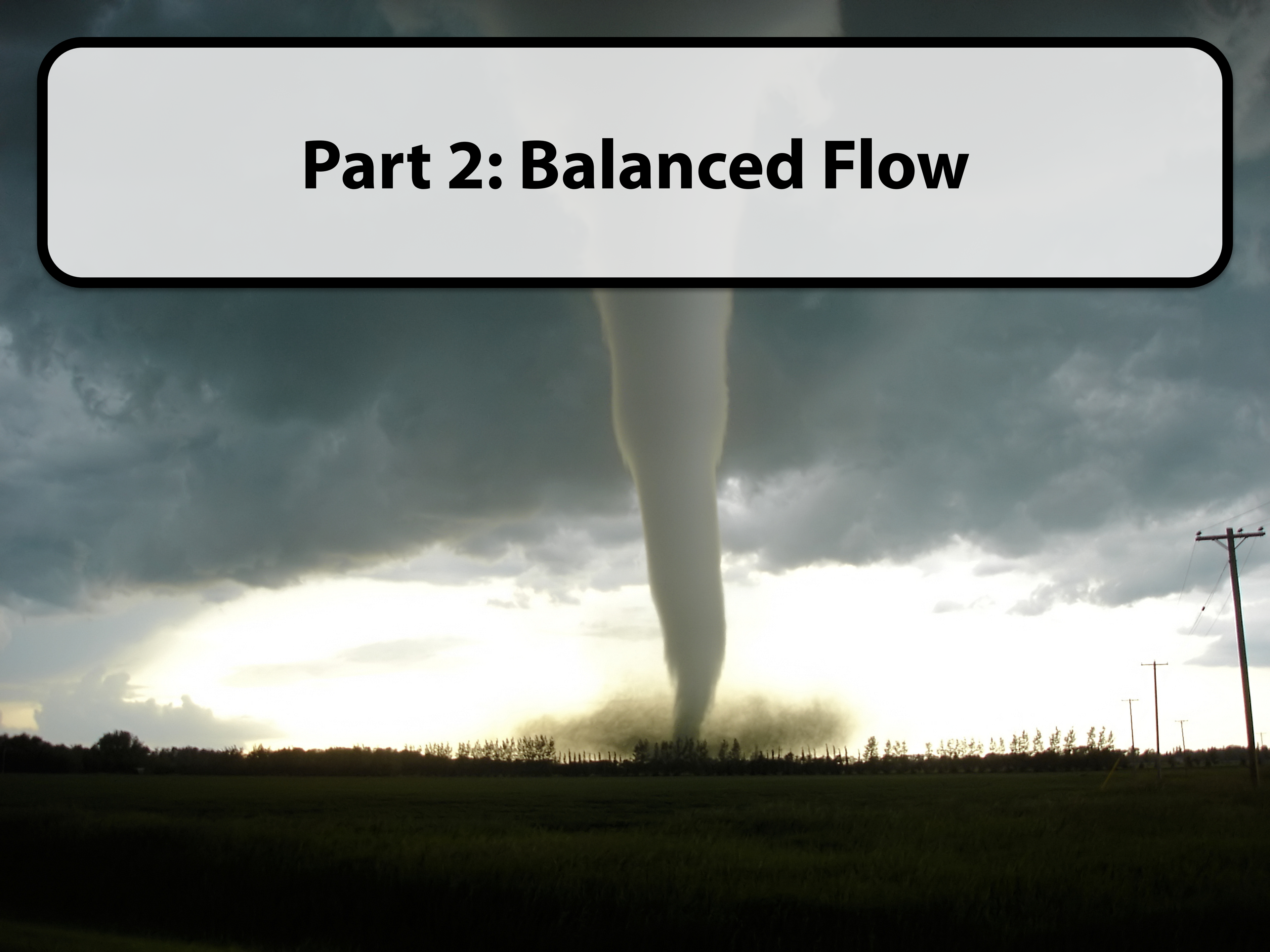
Part 2: Balanced Flow



Momentum Equation

One diagnostic equation for curved flow:

$$\frac{V^2}{R} + fV = -\frac{\partial \Phi}{\partial n}$$

Centrifugal force

Coriolis force

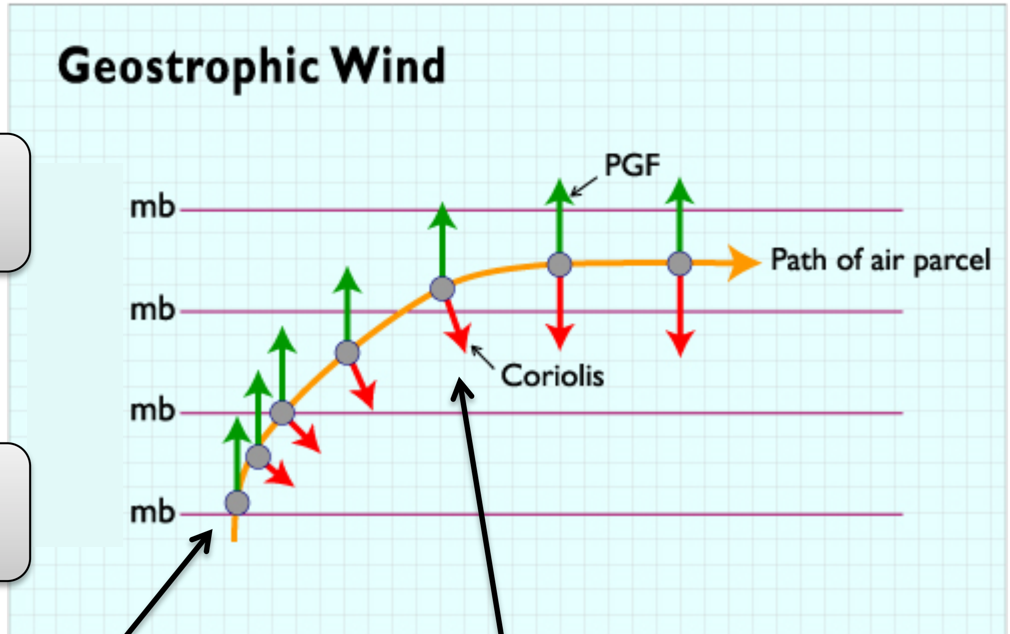
Pressure gradient force

Question: How does this generalize the geostrophic approximation?

Geostrophic Balance

Low Pressure

High Pressure



Flow initiated by pressure gradient

Flow turned by Coriolis force

Geostrophic Balance

Geostrophic balance: Coriolis and pressure gradient force in exact balance (equal magnitude, opposite direction).

Flow is parallel to contours: Flow follows a straight line, so the radius of curvature (R) becomes infinite.

$$\cancelto{0}{\frac{V^2}{R}} + fV = -\frac{\partial \Phi}{\partial n}$$

Centrifugal force

Coriolis force

Pressure gradient force

Geostrophic Balance

In Natural Coordinates

Geostrophic balance in natural coordinates

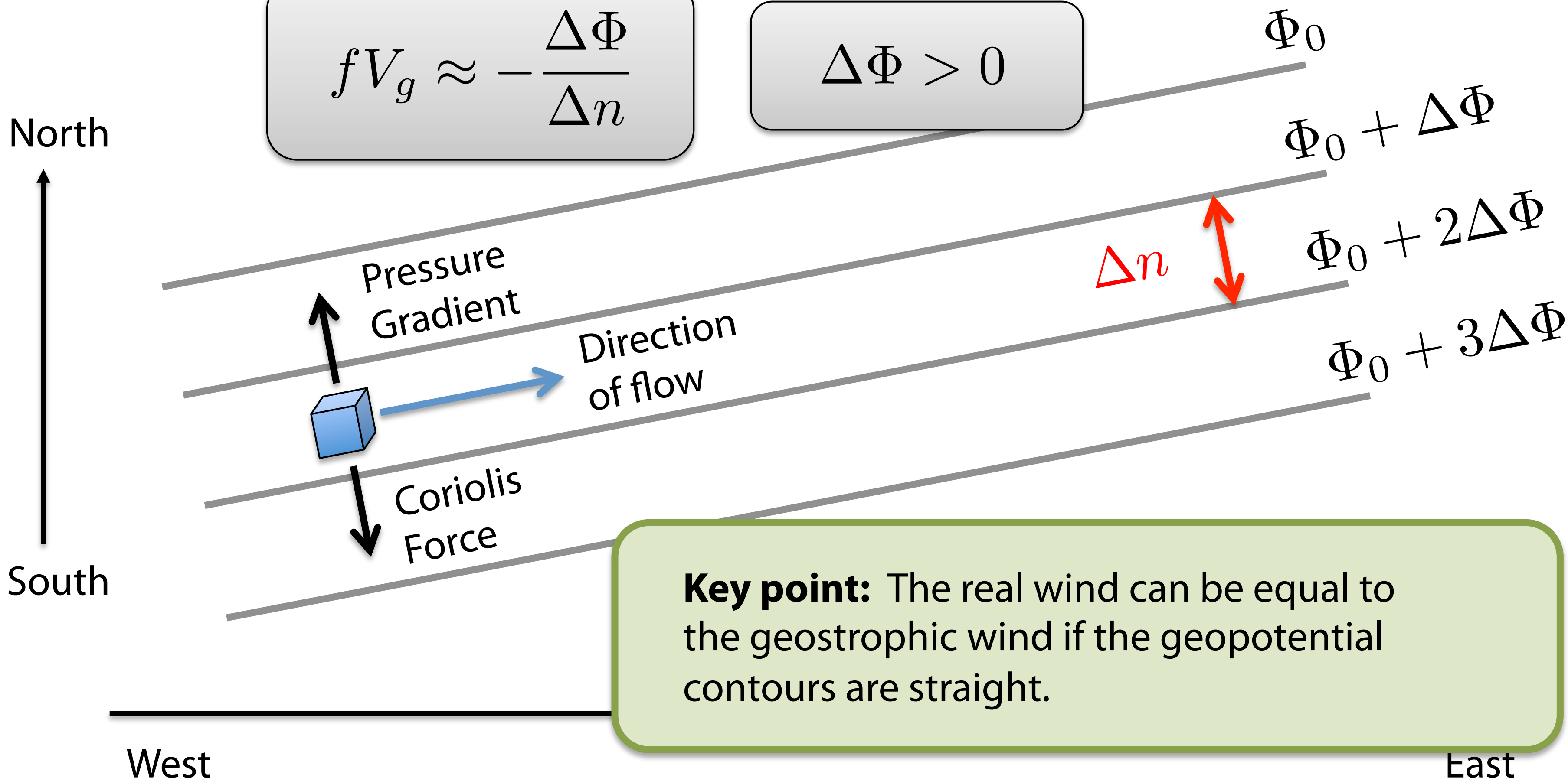
$$fV_g = -\frac{\partial \Phi}{\partial n}$$

Geostrophic Balance

In Natural Coordinates

$$fV_g \approx -\frac{\Delta\Phi}{\Delta n}$$

$$\Delta\Phi > 0$$



Key point: The real wind can be equal to the geostrophic wind if the geopotential contours are straight.

Geostrophic Balance

In Natural Coordinates

Key point: If contours are not straight then the real wind is not geostrophic.

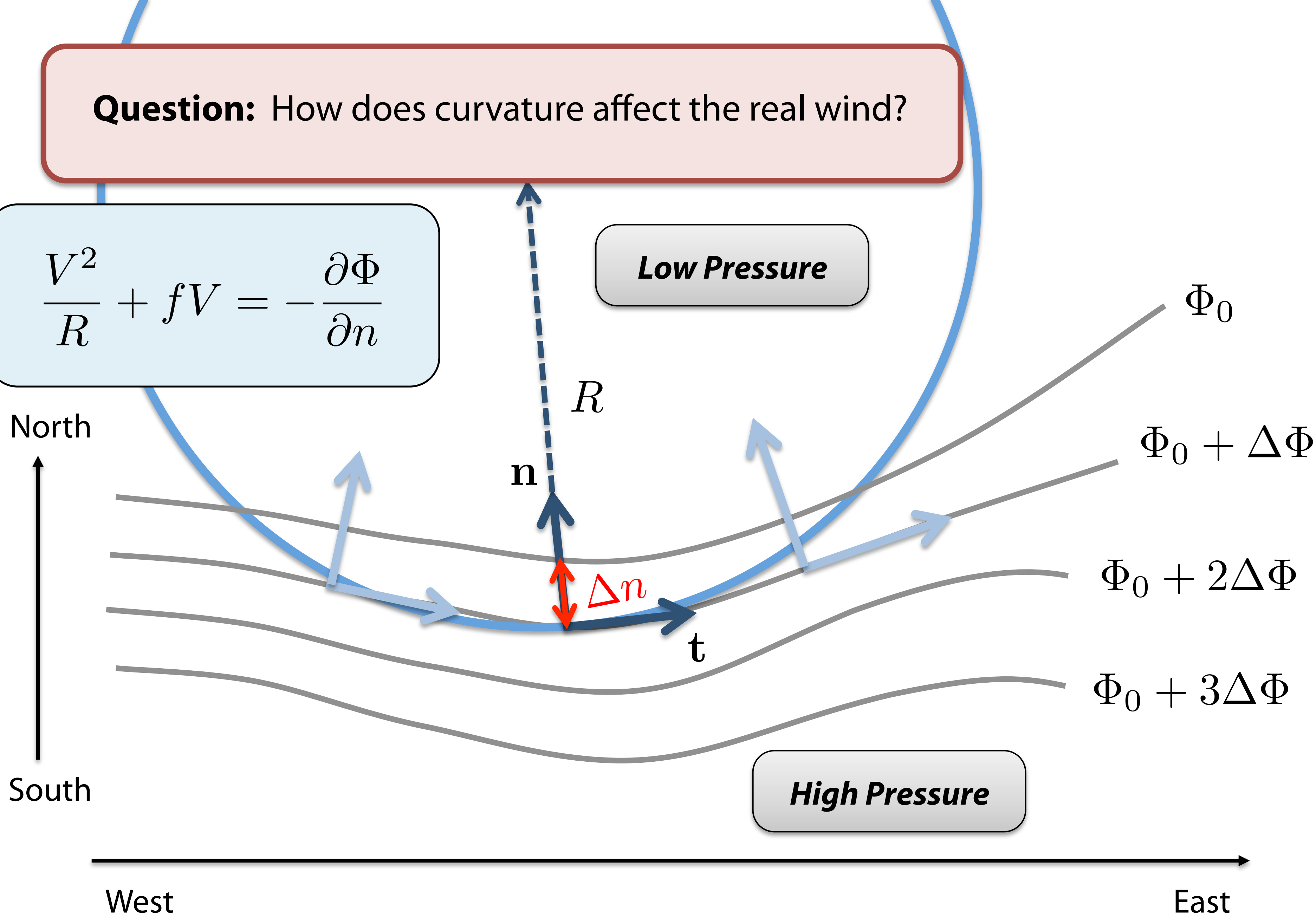
Let's look at this closer…

Definition: Cyclonic flow refers to flow around a low pressure system (geopotential minimum).

Anticyclonic flow refers to flow around a high pressure system (geopotential maximum).

Question: How does curvature affect the real wind?

$$\frac{V^2}{R} + fV = -\frac{\partial \Phi}{\partial n}$$



Ageostrophic Wind

Equation of Motion
(Natural Coordinates)

$$\frac{V^2}{R} + fV = -\frac{\partial \Phi}{\partial n}$$

Split Coriolis into geostrophic and ageostrophic

$$\frac{V^2}{R} + f(V_g + V_{ag}) = -\frac{\partial \Phi}{\partial n}$$

Use definition of geostrophic wind

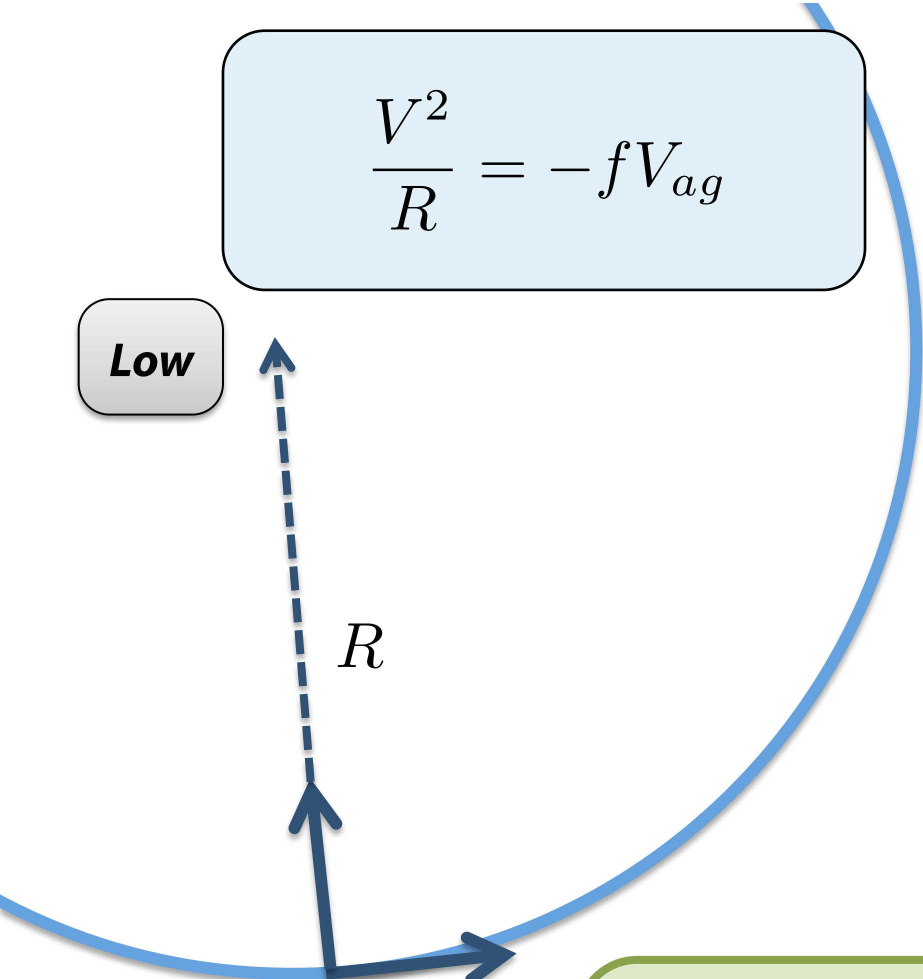
$$fV_g = -\frac{\partial \Phi}{\partial n}$$

$$\frac{V^2}{R} = -fV_{ag}$$

Key point: Centrifugal force balances with ageostrophic part of Coriolis force.

Ageostrophic Wind

$$\frac{V^2}{R} = -fV_{ag}$$



Assume $R > 0$. Then the direction of curvature is towards the low. This is **cyclonic motion.**

$$\frac{V^2}{R} = -fV_{ag}$$

Sign: **> 0** ⇨ **> 0**

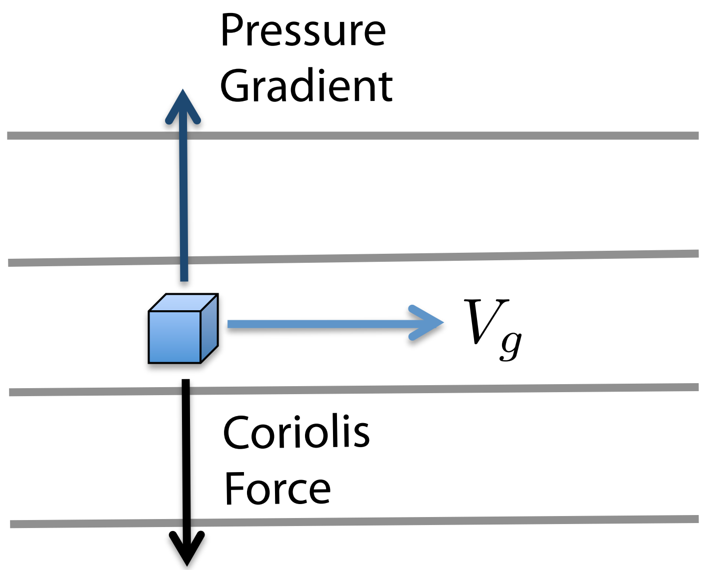
⇨ $V_{ag} < 0$

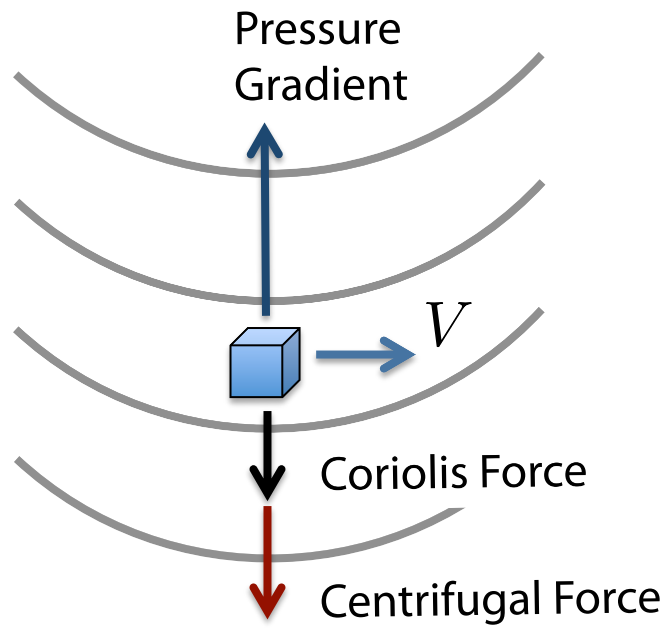
But since $V = V_g + V_{ag}$

⇨ $V < V_g$

Real wind is **slower** than geostrophic wind around a low!

Physical Perspective





No Curvature

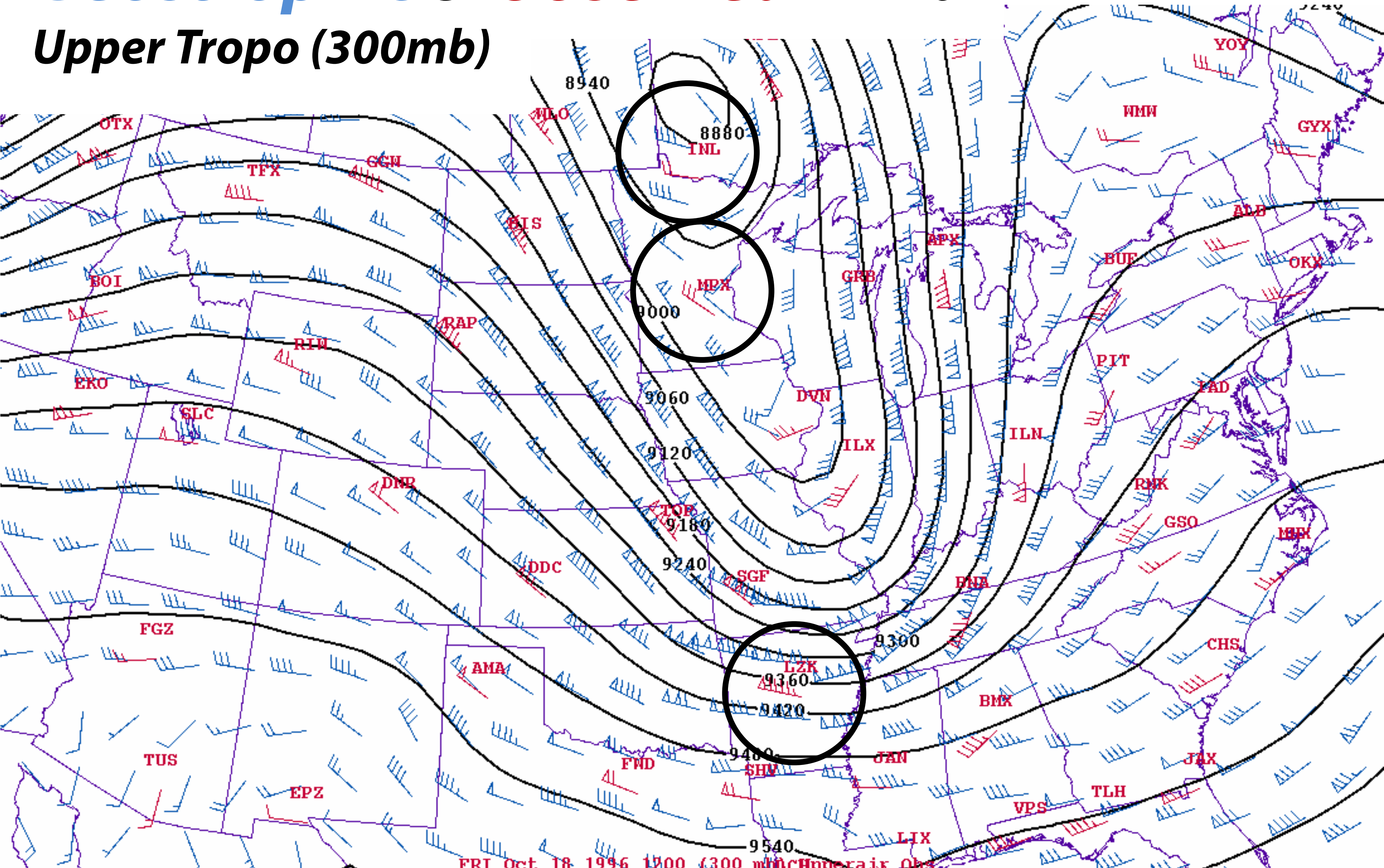
$$V_g > V$$

With Curvature

Pressure gradient is **the same** in each case. However, with curvature less Coriolis force is needed to balance the pressure gradient.

Geostrophic & Observed Wind

Upper Tropo (300mb)

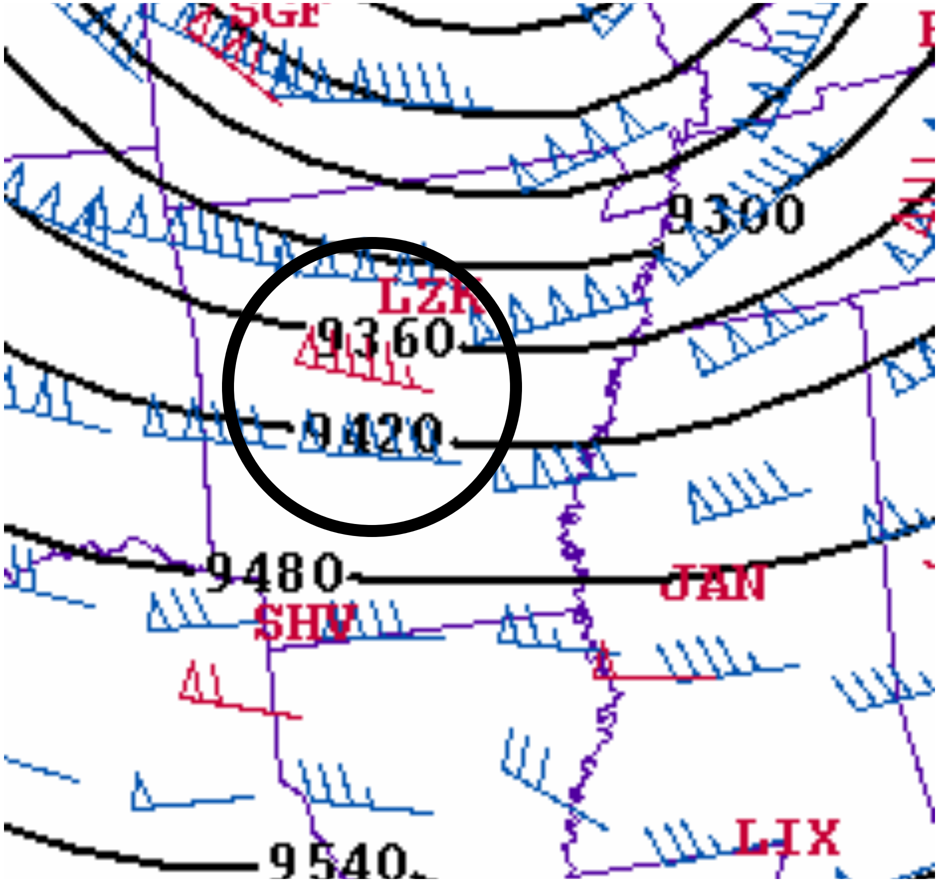


Geostrophic & Observed Wind

Upper Tropo (300mb)

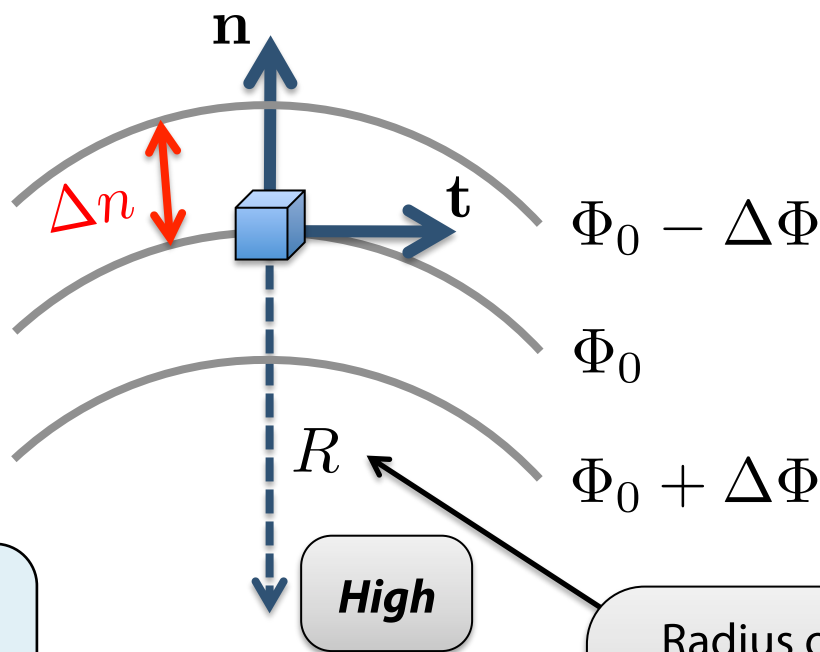
Observed:
95 knots

Geostrophic:
140 knots



Anticyclonic Flow

Flow around a Pressure High



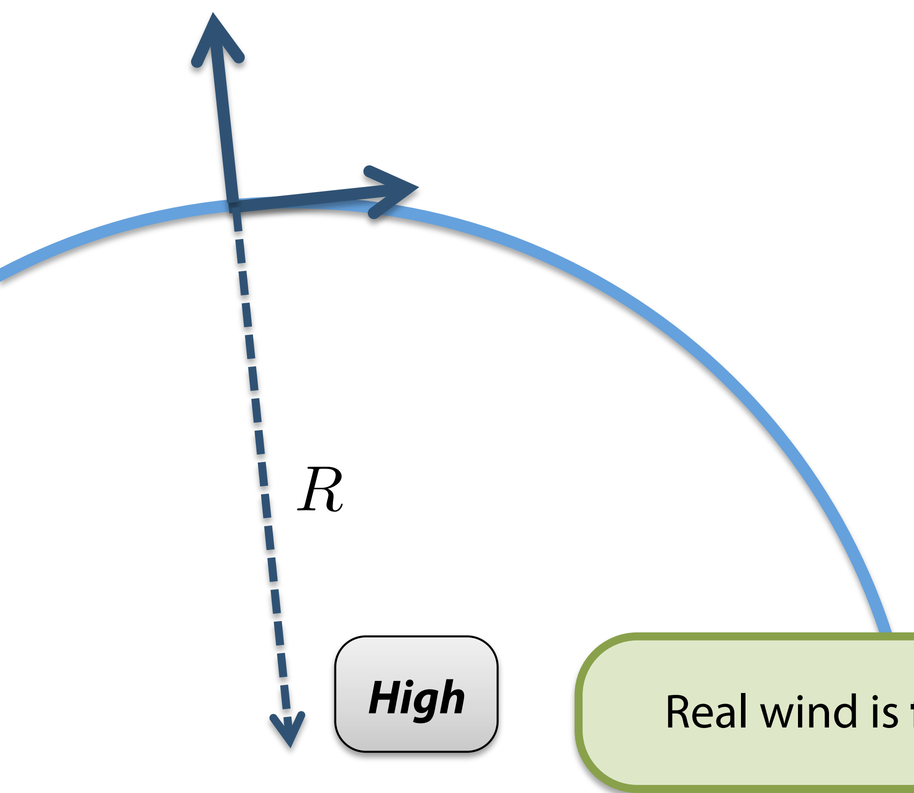
$$\frac{V^2}{R} + fV = -\frac{\partial \Phi}{\partial n}$$

Radius of curvature points in the opposite direction of $\mathbf{n}$

R is negative

Ageostrophic Wind

$$\frac{V^2}{R} = -fV_{ag}$$



Assume $R < 0$. Then the direction of curvature is towards the high. This is **anti-cyclonic motion.**

$$\frac{V^2}{R} = -fV_{ag}$$

Sign: < 0 ⇨ < 0

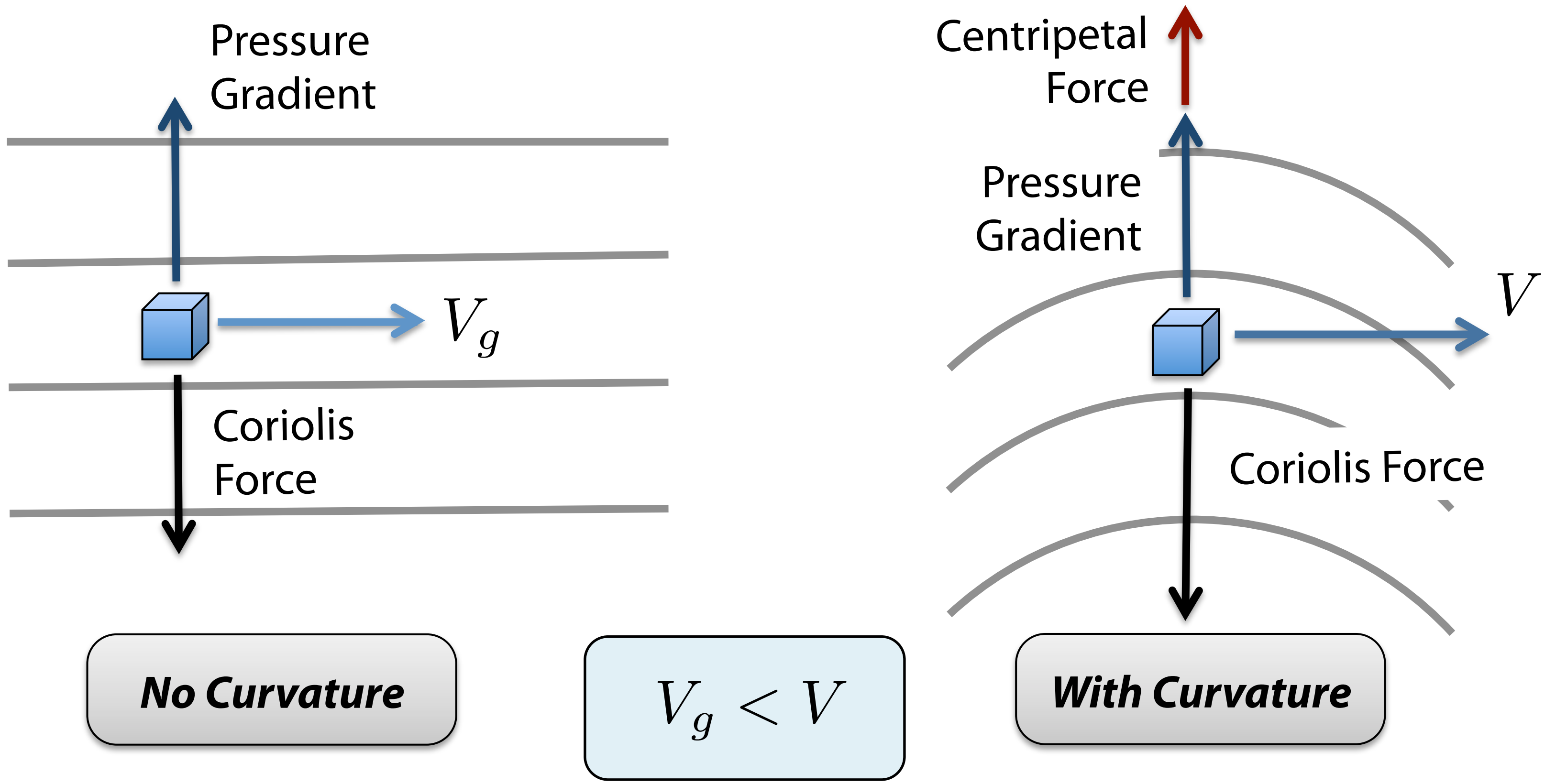
⇨ $V_{ag} > 0$

But since $V = V_g + V_{ag}$

⇨ $V > V_g$

Real wind is **faster** than geostrophic wind around a high!

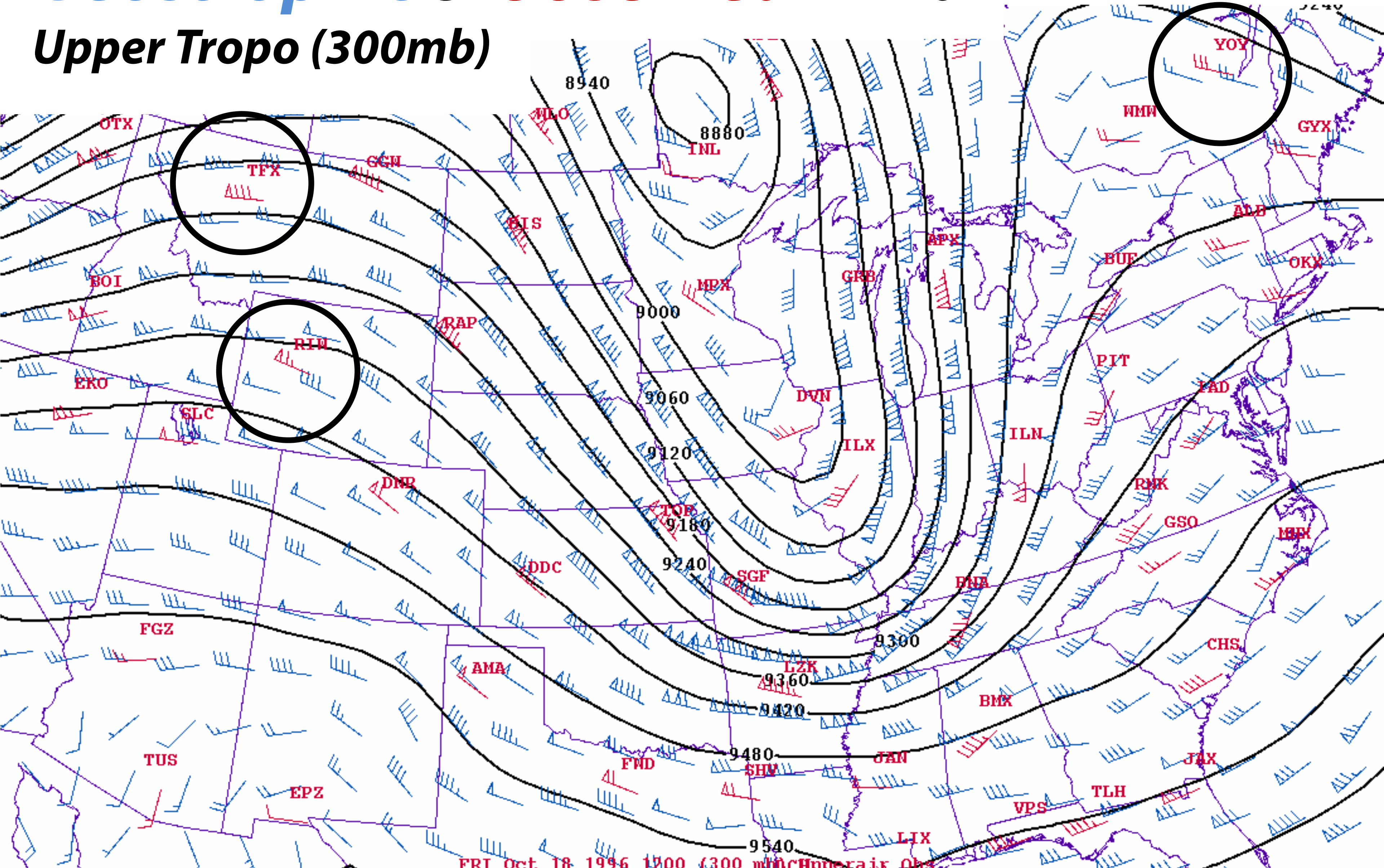
Physical Perspective



Pressure gradient is **the same** in each case. However, with curvature more Coriolis force is needed to balance the pressure gradient + centripetal force.

Geostrophic & *Observed* *Wind*

Upper Tropo (300mb)

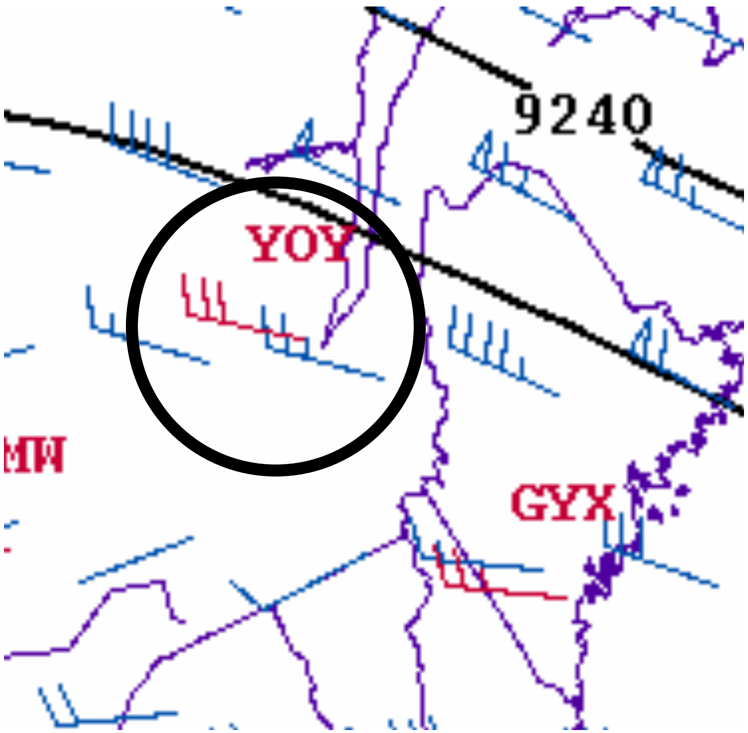


Geostrophic & Observed Wind

Upper Tropo (300mb)

Observed:
30 knots

Geostrophic:
25 knots



Natural Coordinates

Summary

- We found a way to describe balance between pressure gradient force, Coriolis force and curvature (centrifugal force).
  - We assumed friction was unimportant and only looked at flow at a particular level.
    - We assumed flow was on pressure surfaces.
      - We saw that the simplified system can be used to describe real flows in the atmosphere.
        - Can we describe other flow patterns? Different scales? Different regions of the Earth?

Cyclostrophic Flow

Momentum equation in natural coordinates

$$\frac{DV}{Dt}\mathbf{t} + \frac{V^2}{R}\mathbf{n} + fV\mathbf{n} = -\left(\mathbf{t}\frac{\partial \Phi}{\partial s} + \mathbf{n}\frac{\partial \Phi}{\partial n}\right)$$

Momentum equation in natural coordinates, component form

$$\frac{DV}{Dt} = -\frac{\partial \Phi}{\partial s}$$

$$\frac{V^2}{R} + fV = -\frac{\partial \Phi}{\partial n}$$

Question: What about the case of negligible Coriolis force?

Cyclostrophic Flow

Definition: Cyclostrophic flow describes *steady* flows where centrifugal force is balanced by pressure gradient force, and Coriolis force is *largely negligible*.

Cyclostrophic Flow

Normal component of momentum equation

$$\frac{V^2}{R} + fV = -\frac{\partial \Phi}{\partial n}$$

Cyclostrophic flow arises when V is large or R is small. In either case, the centrifugal force term is much larger than Coriolis force.

Question: When might these conditions occur?

Cyclostrophic Flow

Cyclostrophic flow arises when V is large or R is small. In either case, the centrifugal force term is much larger than Coriolis force.

Tornadoes: 100 meter radius, winds up to 50 m/s

Dust Devils: 1 – 10 meter radius, winds up to 25 m/s

Both of these phenomena feature small radii and (relatively) strong winds.

Cyclostrophic Flow

Normal component of momentum equation

$$\frac{V^2}{R} + \cancelto{0}{fV} = -\frac{\partial \Phi}{\partial n}$$

Solve for V: $$V^2 = -R\frac{\partial \Phi}{\partial n} \quad \Rightarrow \quad V = \sqrt{-R\frac{\partial \Phi}{\partial n}}$$

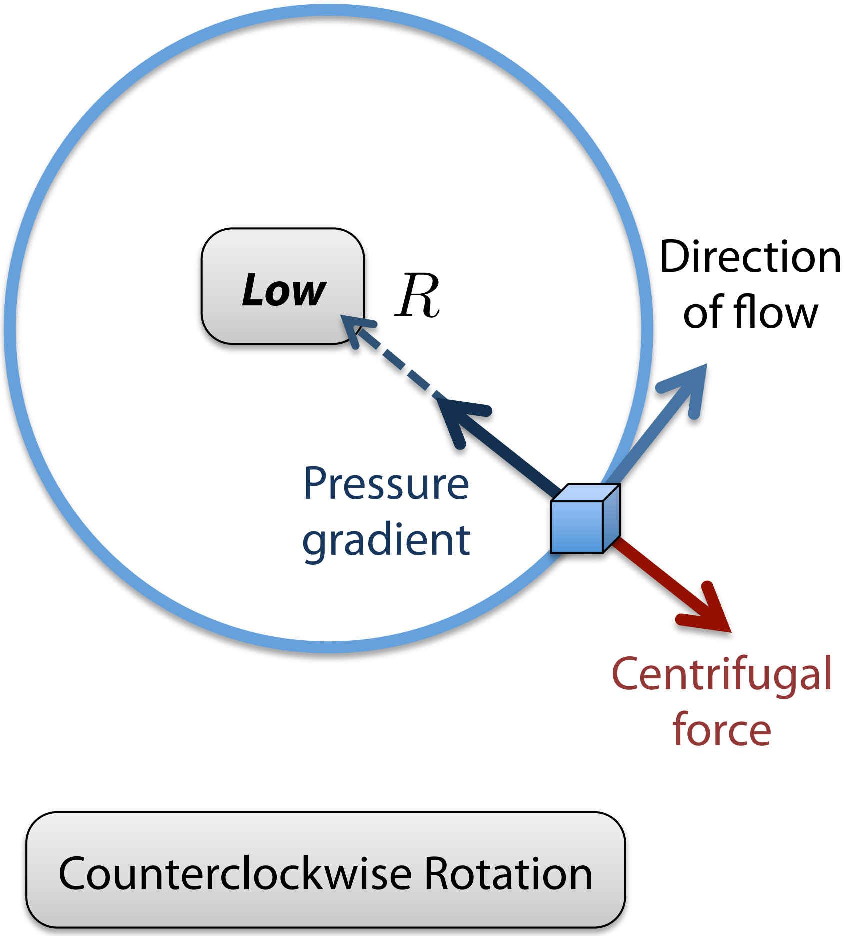
The interior of the square root must be positive (no imaginary winds). This implies two solutions:

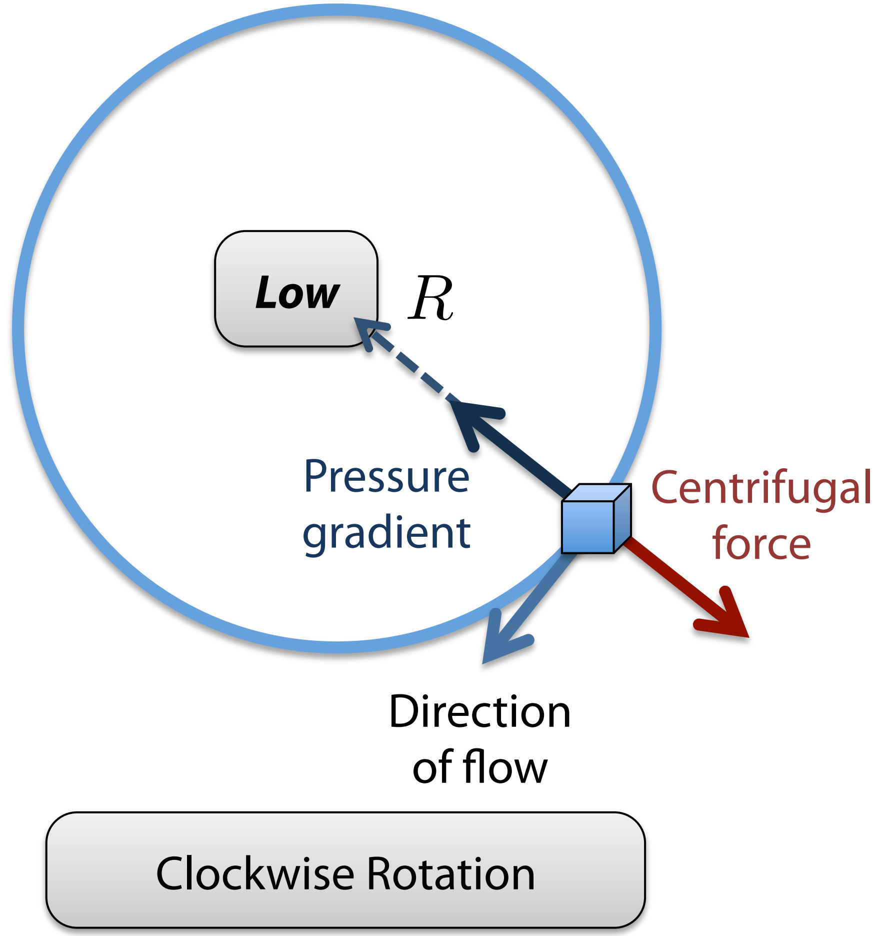
(1) $R > 0, \frac{\partial \Phi}{\partial n} < 0$ **(2)** $R < 0, \frac{\partial \Phi}{\partial n} > 0$

Cyclostrophic Flow

(1) $R > 0, \dfrac{\partial \Phi}{\partial n} < 0$

(2) $R < 0, \dfrac{\partial \Phi}{\partial n} > 0$

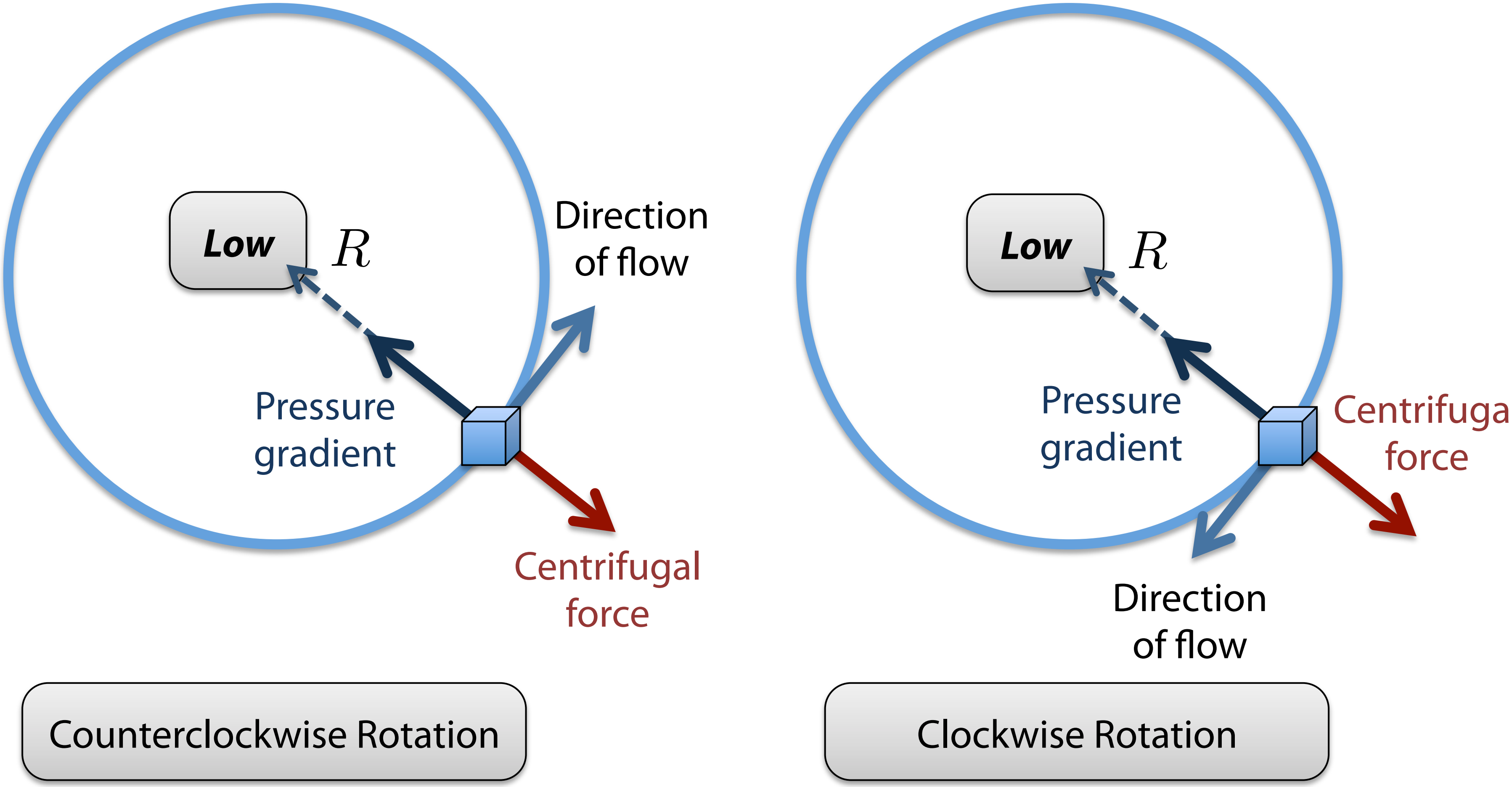




Cyclostrophic Flow

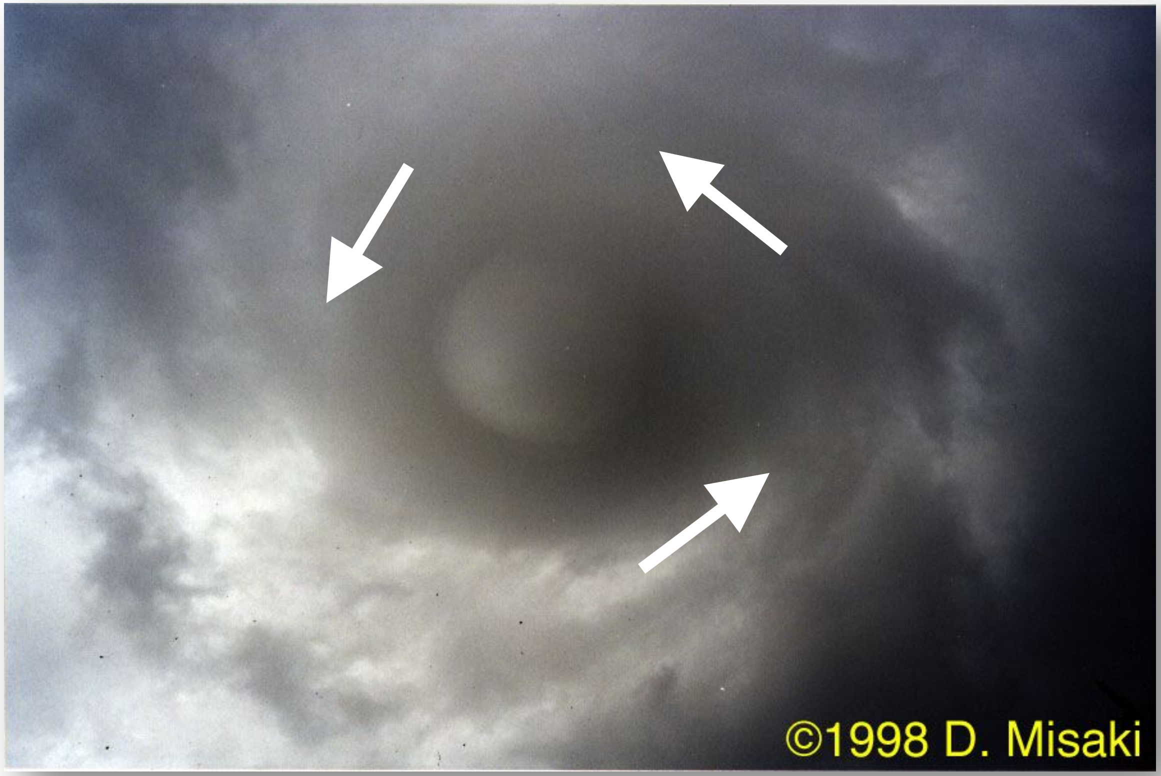
(1) $R > 0, \dfrac{\partial \Phi}{\partial n} < 0$

(2) $R < 0, \dfrac{\partial \Phi}{\partial n} > 0$



Anticyclonic Tornado

Looking up

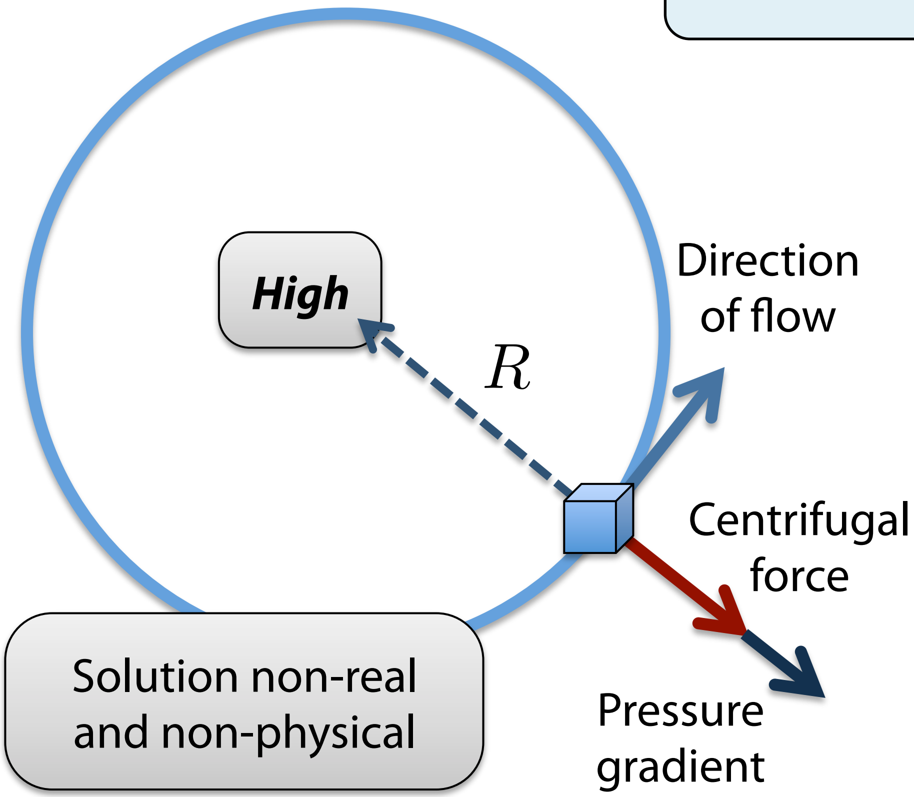


Sunnyvale, CA 4 May 1998

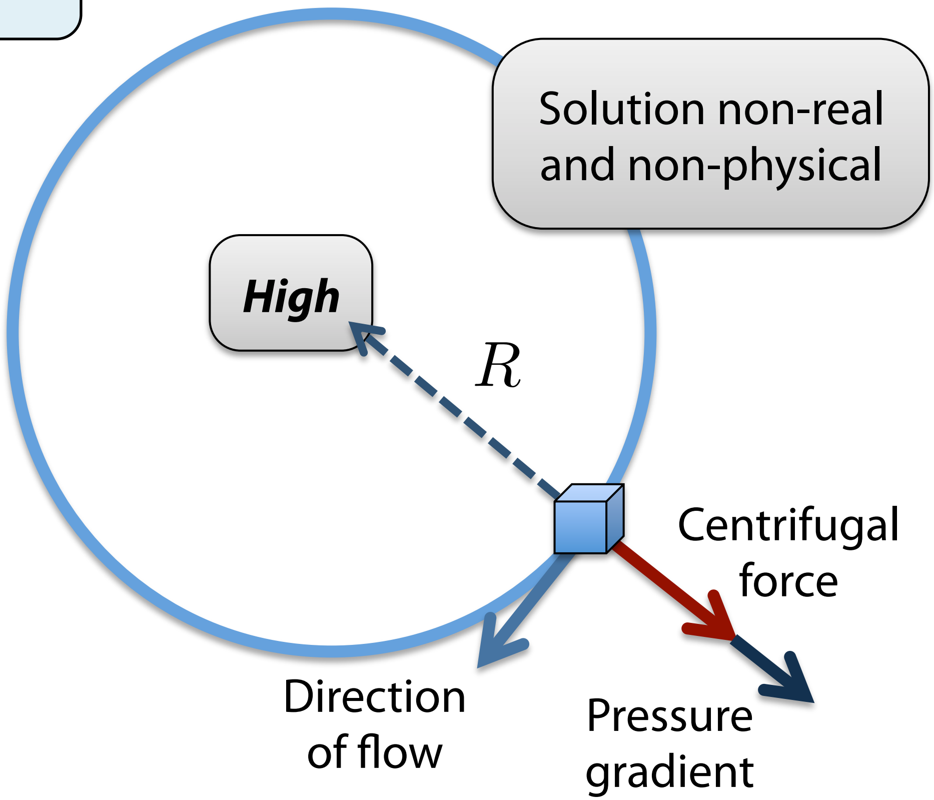
Question: Why can't we have cyclostrophic flow around a high pressure system?

$$V = \sqrt{-R\frac{\partial\Phi}{\partial n}}$$

(1) $R > 0, \dfrac{\partial\Phi}{\partial n} > 0$



(2) $R < 0, \dfrac{\partial\Phi}{\partial n} < 0$



Inertial Flow

Definition: Inertial flow describes *steady* flows where centrifugal force is balanced by Coriolis force and pressure gradients are *largely negligible*.

$$\frac{V^2}{R} + fV = -\cancel{\frac{\partial \Phi}{\partial n}}^{\,0}$$

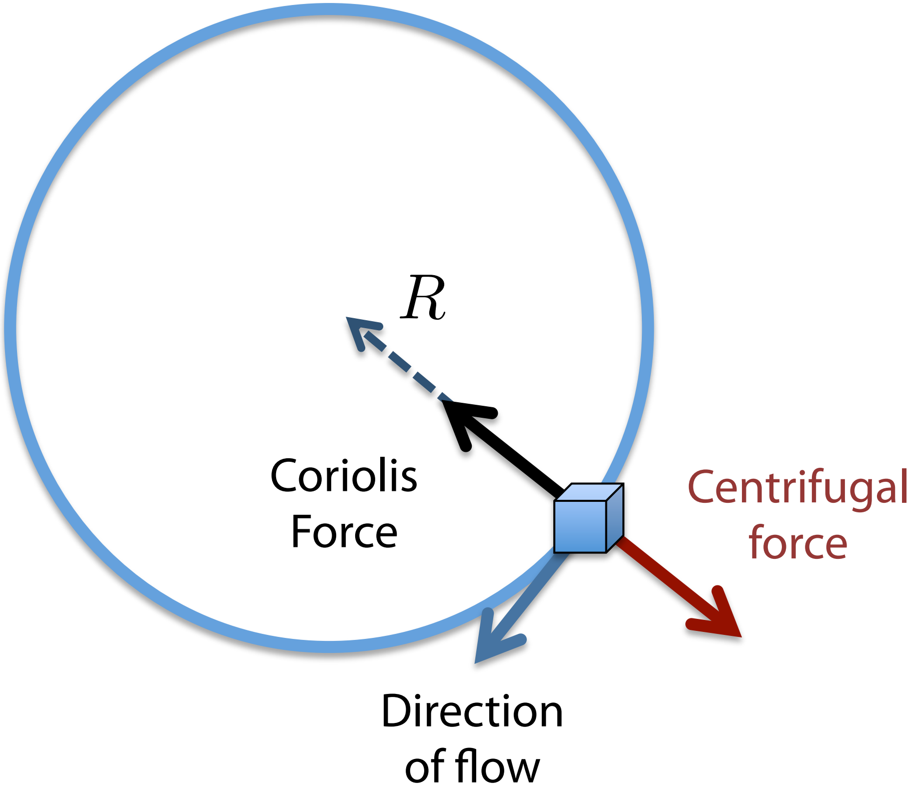
Solutions $\cancel{V = 0}$ Uninteresting

and $\boxed{V = -fR}$

Since $V > 0$, must have $R < 0$

Inertial Flow

$$\boxed{V = -fR}$$



Inertial motion is always anti-cyclonic (clockwise in northern hemisphere).

Since the circumference of the circle is

$$C = 2\pi R$$

The period of rotation is

$$P = \left|\frac{C}{V}\right| = \frac{2\pi R}{fR} = \frac{\pi}{\Omega|\sin\phi|}$$

But since Ω is rotation rate of the Earth

$$\boxed{P = \frac{\frac{1}{2}\ \mathrm{day}}{|\sin\phi|}}$$

Gradient Flow

Momentum equation in natural coordinates, component form

$$\frac{DV}{Dt} = -\frac{\partial \Phi}{\partial s}$$

$$\frac{V^2}{R} + fV = -\frac{\partial \Phi}{\partial n}$$

So far we have investigated:

- Balance between pressure gradient force and Coriolis (**Geostrophic Balance**).
- Balance between pressure gradient force and Centrifugal force (**Cyclostrophic flow**).
- Balance between centrifugal force and Coriolis force (**Inertial flow**).
- We now consider balance between all three terms in the normal momentum equation (**Gradient flow**).

Gradient Flow

$$\frac{V^2}{R} + fV = -\frac{\partial \Phi}{\partial n}$$

Observe this is simply a quadratic equation in V

$$V^2 + fRV + R\frac{\partial \Phi}{\partial n} = 0$$

Quadratic Equation

$$V = \frac{1}{2}\left[-fR \pm \sqrt{(-fR)^2 - 4R\frac{\partial \Phi}{\partial n}}\right]$$

$$\boxed{V = -\frac{fR}{2} \pm \sqrt{\frac{(fR)^2}{4} - R\frac{\partial \Phi}{\partial n}}}$$

Gradient flow velocity

Gradient Flow

The solution V must be **real** and **non-negative**.

$$-\frac{fR}{2} + \sqrt{\frac{(fR)^2}{4} - R\frac{\partial\Phi}{\partial n}}$$

$\cancel{\frac{\partial\Phi}{\partial n} > 0, R > 0}$ $\quad V < 0$

$\frac{\partial\Phi}{\partial n} > 0, R < 0$ **Anomalous Low**

$\frac{\partial\Phi}{\partial n} < 0, R > 0$ **Normal Low**

$\frac{\partial\Phi}{\partial n} < 0, R < 0$ **Anomalous High**

$$-\frac{fR}{2} - \sqrt{\frac{(fR)^2}{4} - R\frac{\partial\Phi}{\partial n}}$$

$\cancel{\frac{\partial\Phi}{\partial n} > 0, R > 0}$ $\quad V < 0$

$\cancel{\frac{\partial\Phi}{\partial n} > 0, R < 0}$ $\quad V < 0$

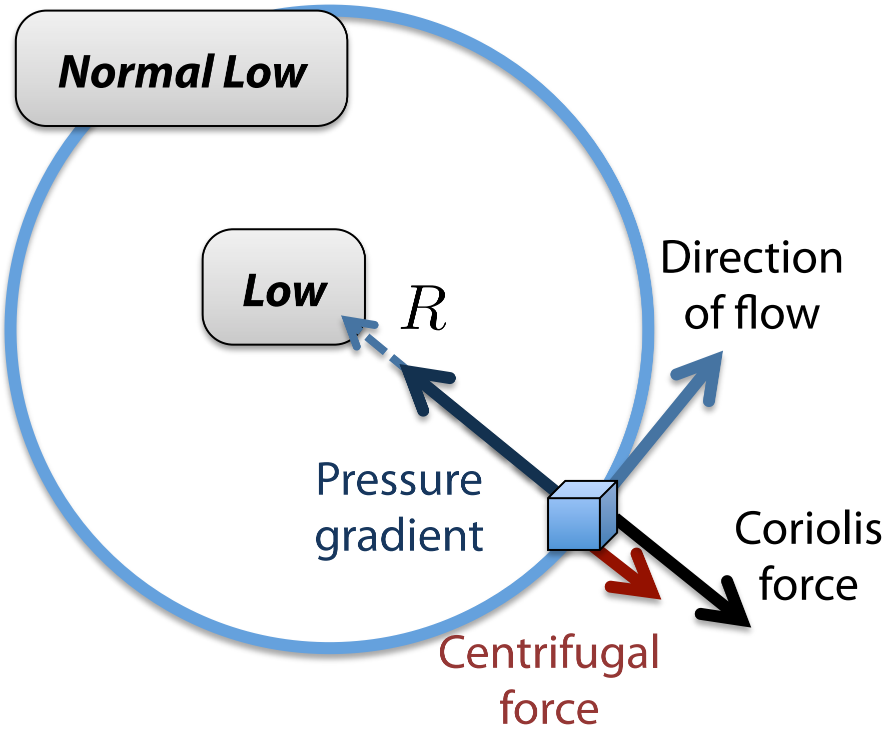
$\cancel{\frac{\partial\Phi}{\partial n} < 0, R > 0}$ $\quad V < 0$

$\boxed{\frac{\partial\Phi}{\partial n} < 0, R < 0}$ **Normal High**

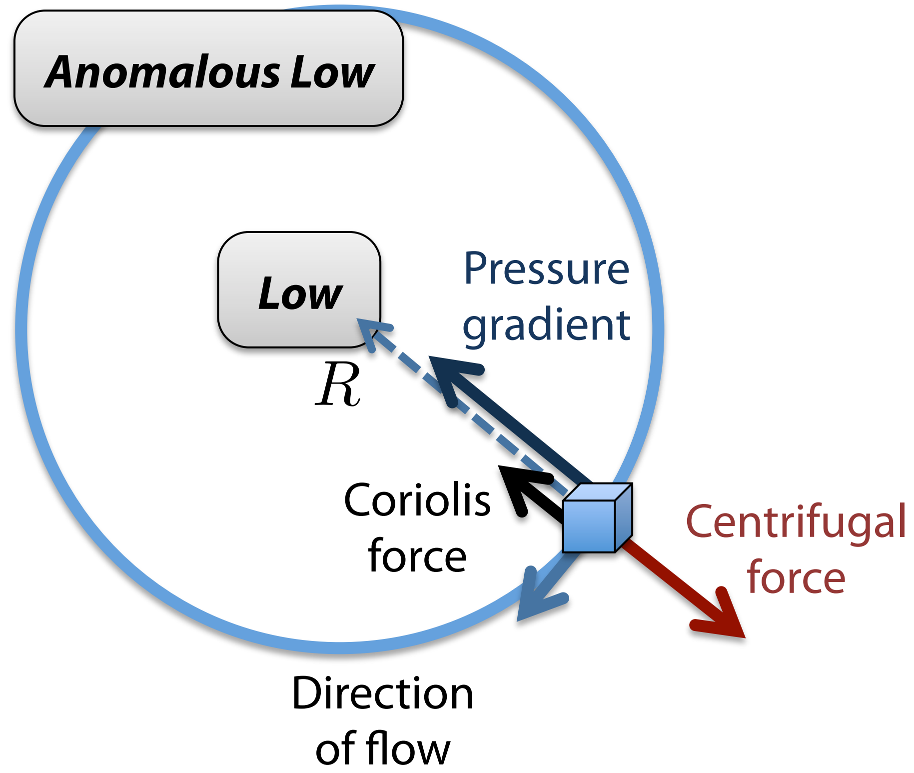
Gradient Flow

Solutions for Lows

(1) $\frac{\partial \Phi}{\partial n} < 0, R > 0$



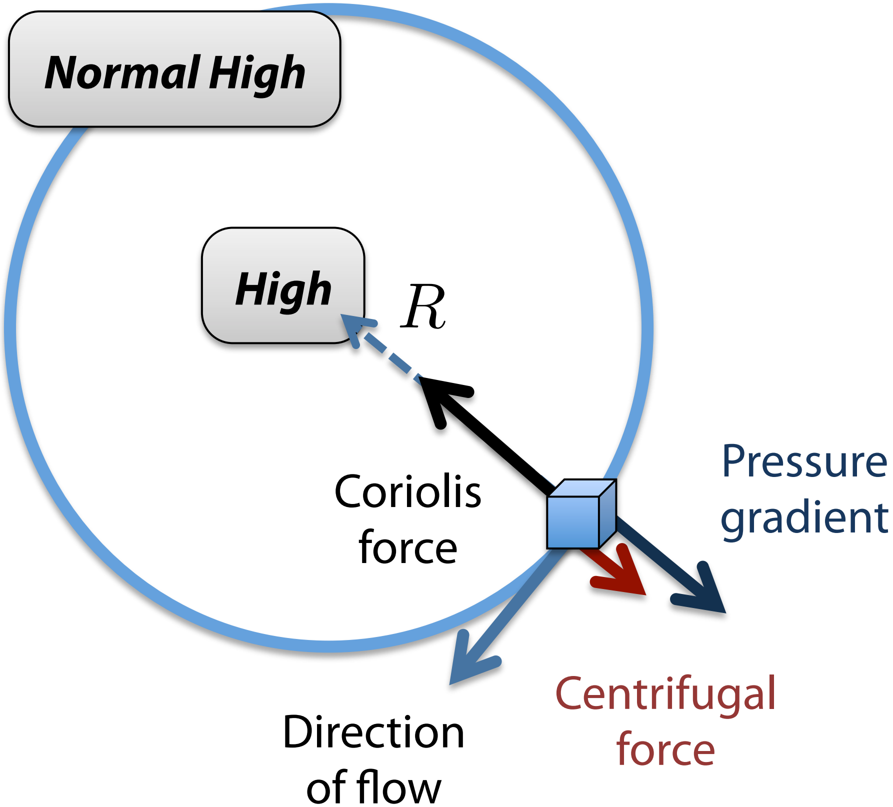
(2) $\frac{\partial \Phi}{\partial n} > 0, R < 0$



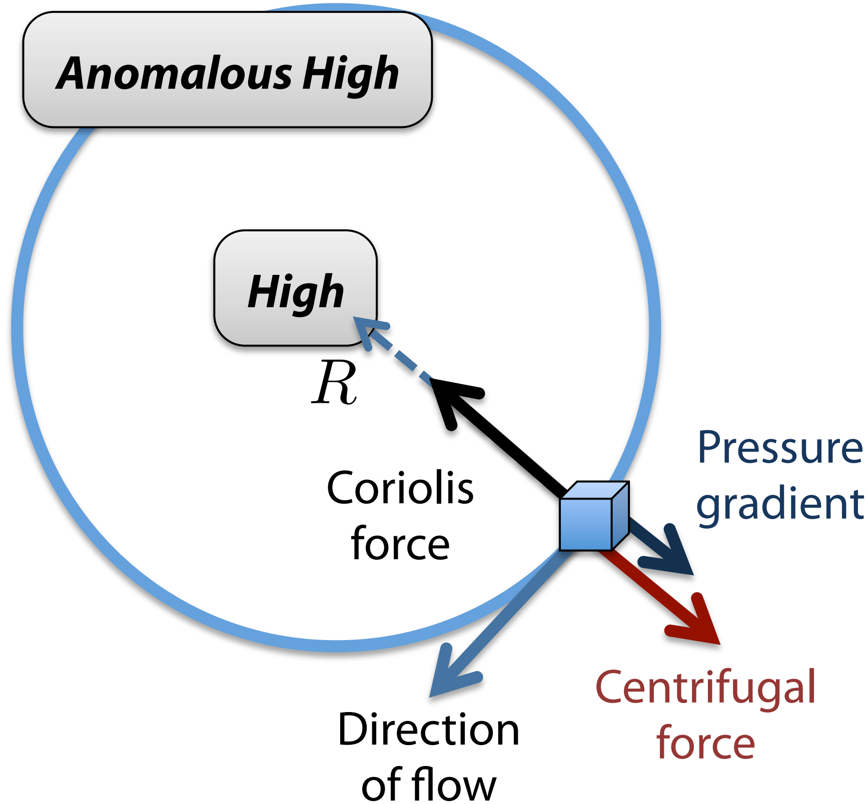
Gradient Flow

Solutions for Highs

(1) $\dfrac{\partial \Phi}{\partial n} < 0, R < 0$



(2) $\dfrac{\partial \Phi}{\partial n} < 0, R < 0$



Gradient Flow

Implications

$$V = -\frac{fR}{2} \pm \underbrace{\sqrt{\frac{(fR)^2}{4} - R\frac{\partial \Phi}{\partial n}}}$$

The discriminant (the term inside the square root) must be positive for a stable solution to exist.

$$\frac{(fR)^2}{4} - R\frac{\partial \Phi}{\partial n} \geq 0$$

$R > 0$

$$\frac{\partial \Phi}{\partial n} \leq \frac{f^2 R}{4}$$

$R < 0$

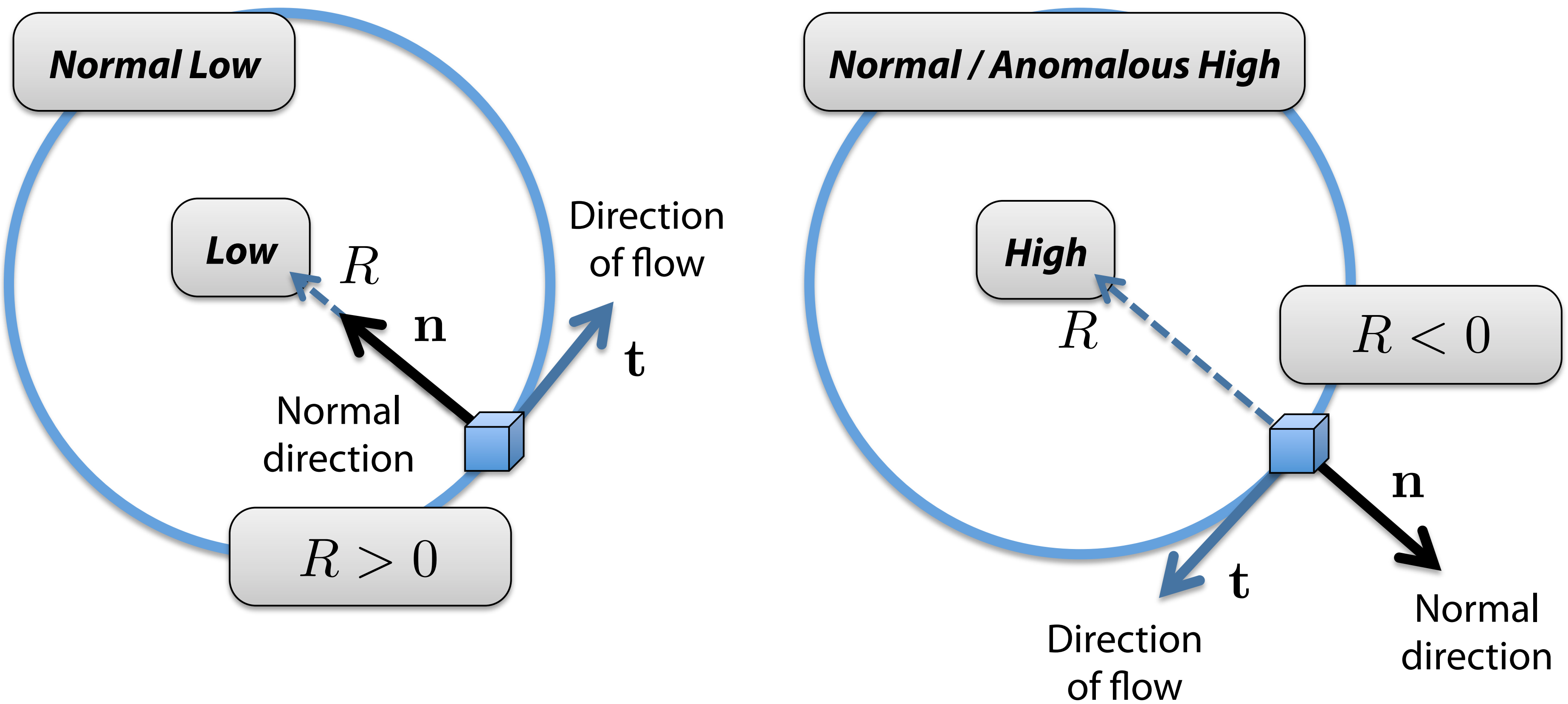
$$\frac{\partial \Phi}{\partial n} \geq \frac{f^2 R}{4}$$

This is a constraint on the pressure gradient that must hold for any **steady** flow.

Gradient Flow

Implications

Consider $\dfrac{\partial \Phi}{\partial n} < 0$



Gradient Flow

Normal Low

$$\frac{\partial \Phi}{\partial n} < 0, R > 0$$

$$\boxed{\frac{\partial \Phi}{\partial n} \leq \frac{f^2 R}{4}}$$

Always satisfied!

Normal / Anomalous High

$$\frac{\partial \Phi}{\partial n} < 0, R < 0$$

$$\boxed{\frac{\partial \Phi}{\partial n} \geq \frac{f^2 R}{4}}$$

Trouble! Constraint implies $\left|\frac{\partial \Phi}{\partial n}\right| \leq \frac{f^2 |R|}{4}$

In anti-cyclonic gradient flow, pressure must go to zero faster than R goes to zero. Hence high pressure regions must have relatively flat pressure gradient.

Gradient Flow

Normal / Anomalous Flows

- **Normal flows** are observed all the time
  - Normal highs (atmospheric blocks) tend to have slower winds than normal lows (tropical cyclones, extratropical cyclones) since the pressure gradient is bounded.
  - Normal lows are storms; normal highs are fair weather.
- **Anomalous flows** are not often observed
  - Anomalous highs have been reported in the tropics.
  - Anomalous lows are "strange" – quoting Holton, "clearly not a useful approximation." However, these do appear for very small values of R (ie. Tornados).