Applications of the Basic Equations
Chapter 3

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Part 1: Natural Coordinates

Natural Coordinates

Question: Why do we need *another* coordinate system?

Our goal is to ***simplify*** the equations of motion. Sometimes complicated equations are simple if looked at in the right way.

At large scales, the atmosphere is in a state of balance. At large scales, mass fields (ρ, p, Φ) balance with wind fields ($\mathbf{u}$).

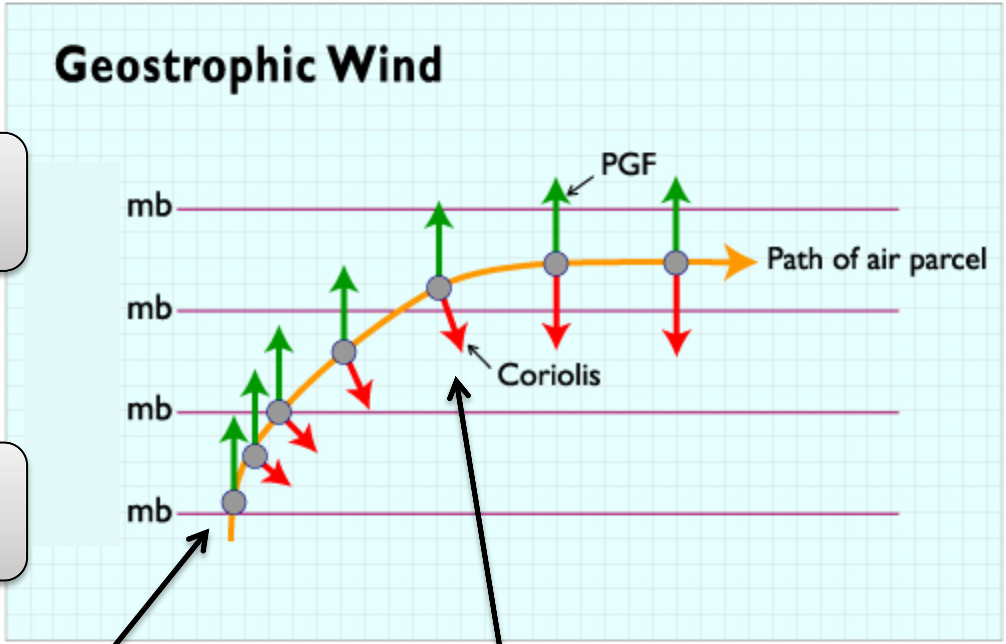
But mass fields are generally much easier to observe than wind.

Balance provides a way to infer the wind from the observed pressure or geopotential.

Geostrophic Balance

Low Pressure

High Pressure

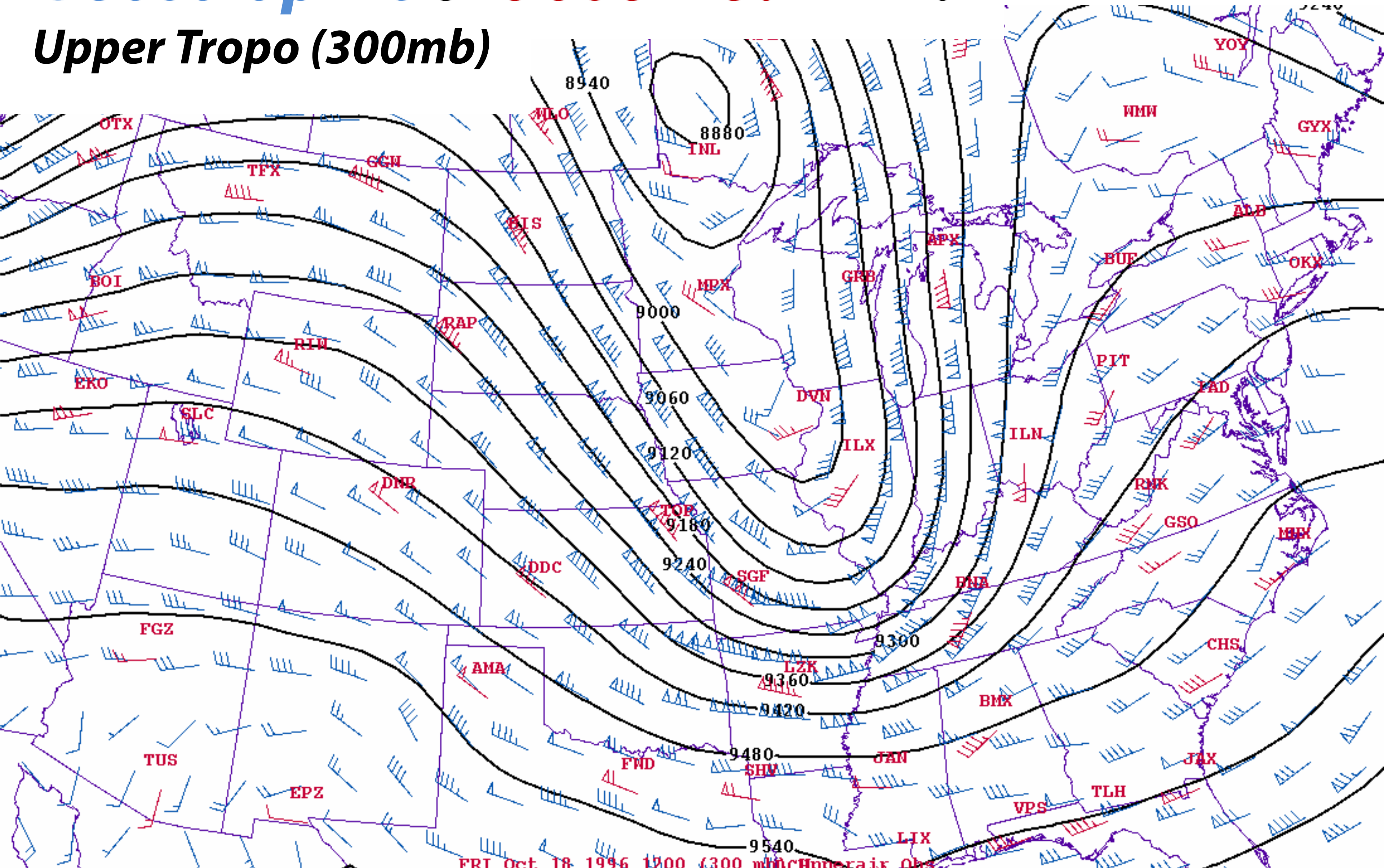


Flow initiated by pressure gradient

Flow turned by Coriolis force

Geostrophic & Observed Wind

Upper Tropo (300mb)



Describe the Previous Figure…

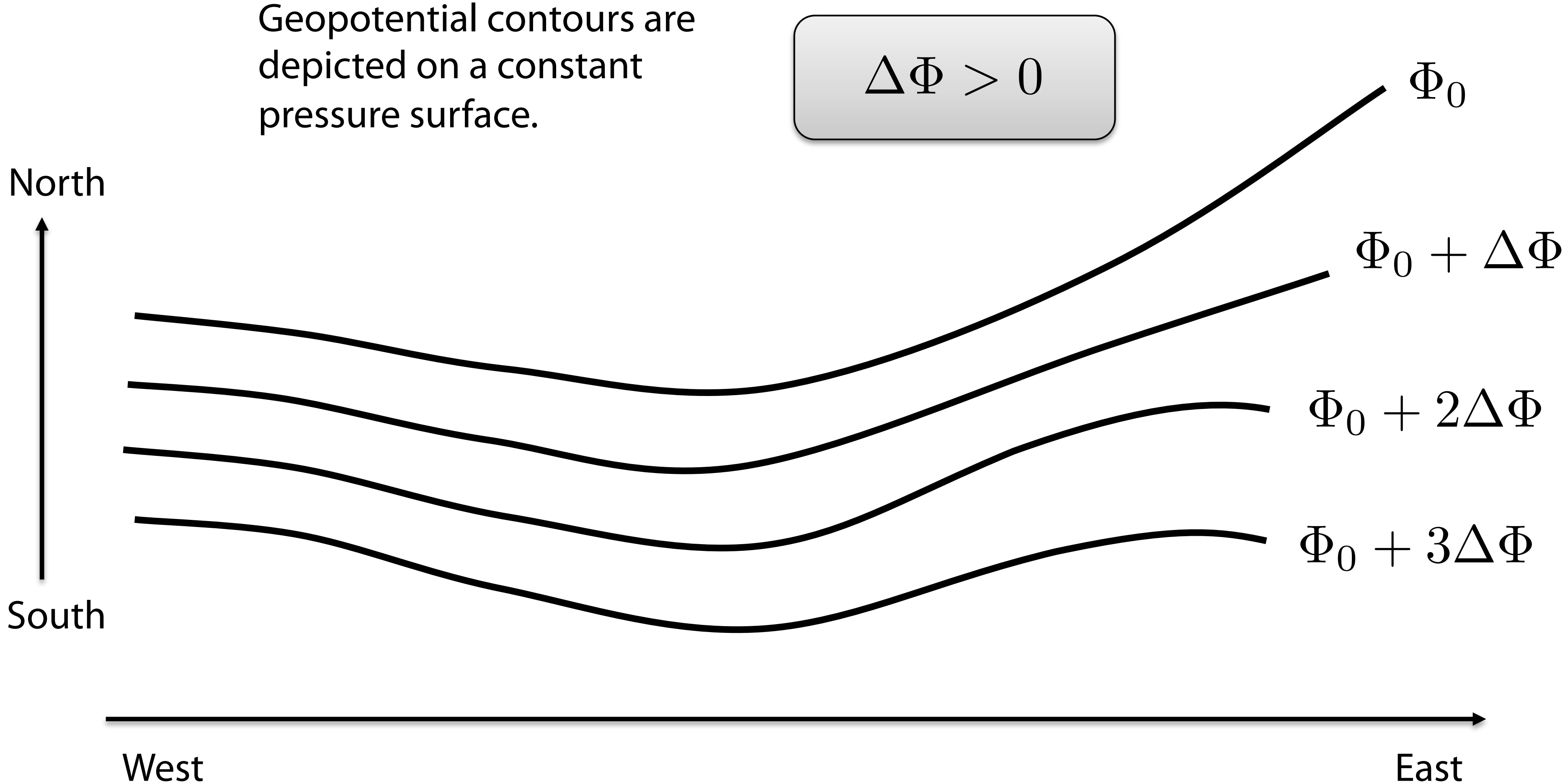
At upper levels (where friction is negligible) the observed wind is parallel to geopotential height contours (on a constant pressure surface).

Wind is ***faster*** when height contours are close together.

Wind is ***slower*** when height contours are farther apart.

The Upper Troposphere

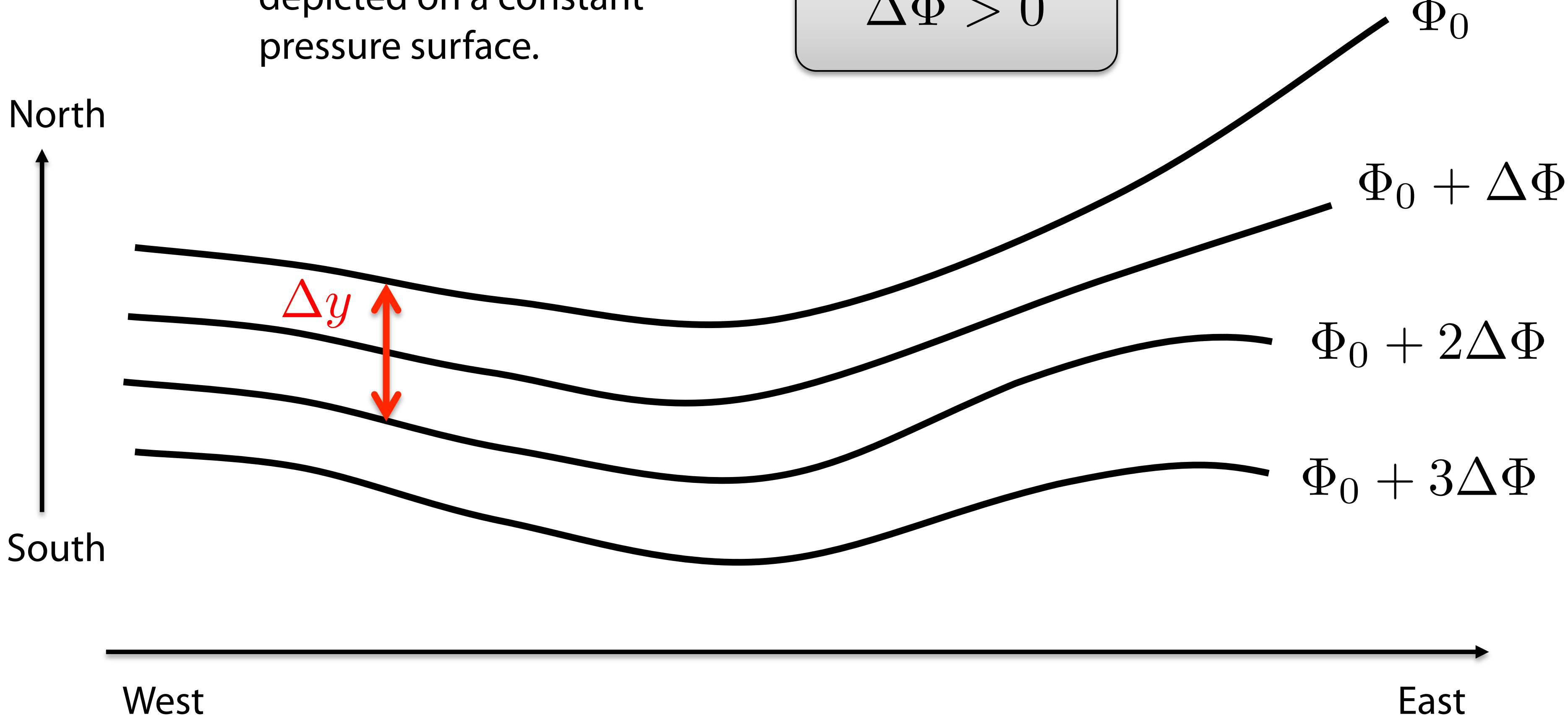
Geopotential contours are depicted on a constant pressure surface.



The Upper Troposphere

Geopotential contours are depicted on a constant pressure surface.

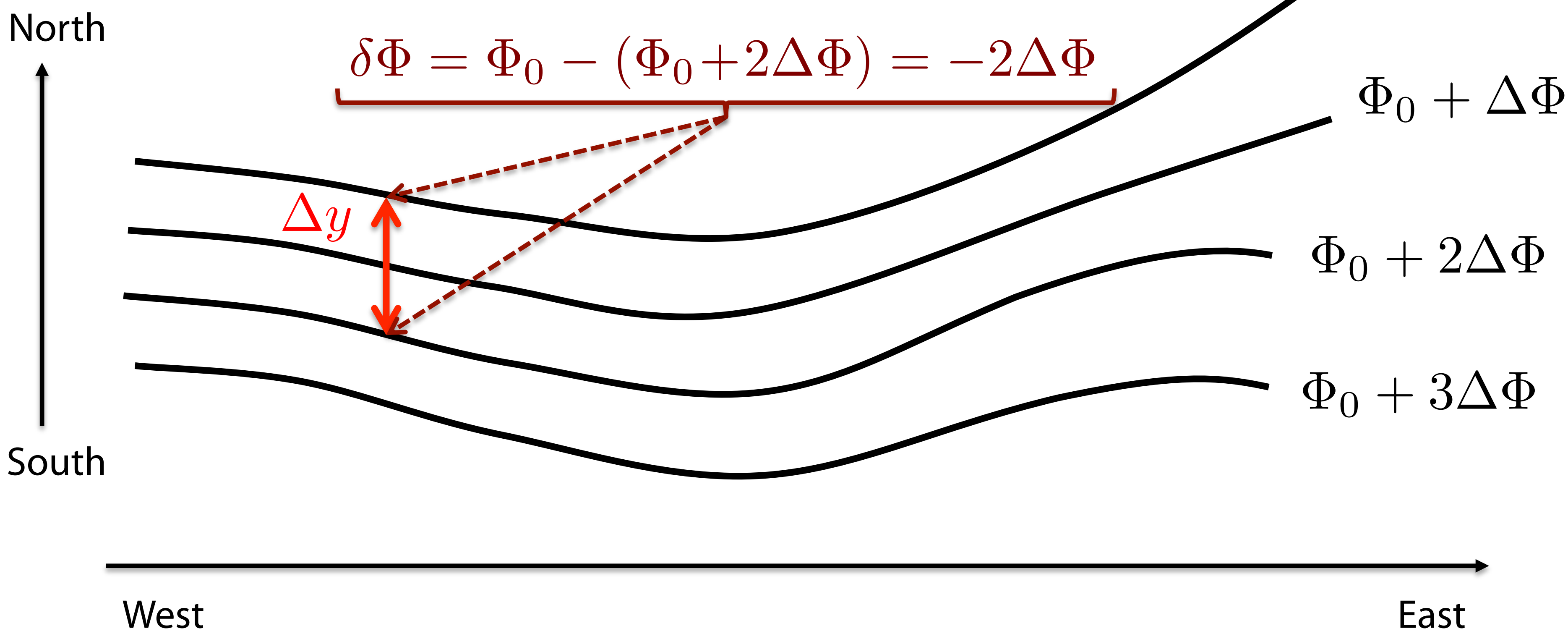
$$\Delta\Phi > 0$$



The Upper Troposphere

Geopotential contours are depicted on a constant pressure surface.

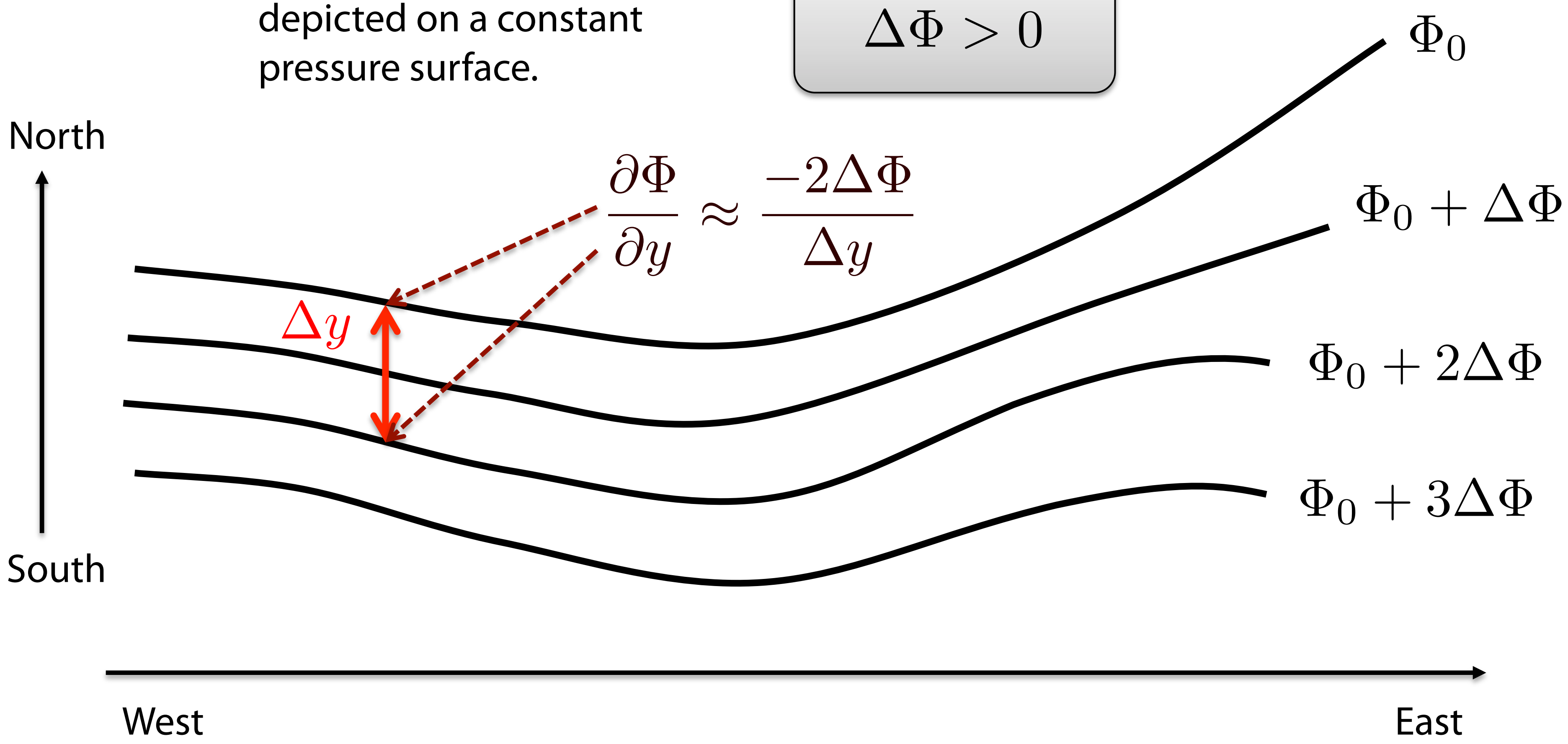
$$\boxed{\Delta\Phi > 0}$$



The Upper Troposphere

Geopotential contours are depicted on a constant pressure surface.

$$\Delta\Phi > 0$$



Horizontal Momentum

Assume no viscosity

$$\left(\frac{d\mathbf{u}}{dt}\right)_p + f\mathbf{k} \times \mathbf{u} = -\nabla_p \Phi$$

$$\left(\frac{du}{dt}\right)_p = -\left(\frac{\partial \Phi}{\partial x}\right)_p + fv$$

$$\left(\frac{dv}{dt}\right)_p = -\boxed{\left(\frac{\partial \Phi}{\partial y}\right)_p} - fu$$

Meridional gradient of
geopotential appears here

Geostrophic Approximation

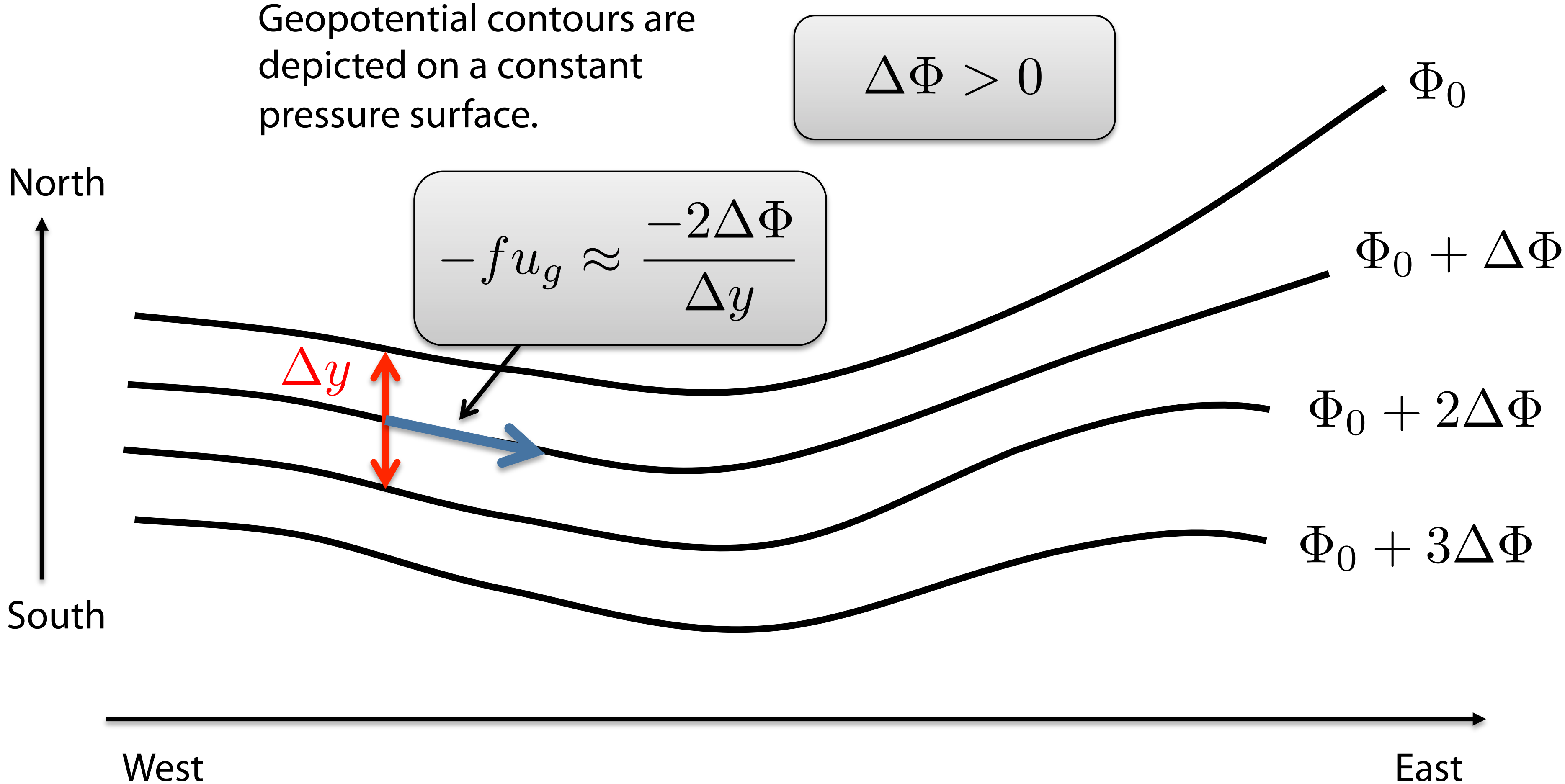
$$\left(\frac{\partial \Phi}{\partial x}\right)_p = f v_g$$

$$-\boxed{\left(\frac{\partial \Phi}{\partial y}\right)_p} = f u_g$$

Meridional gradient of geopotential appears here

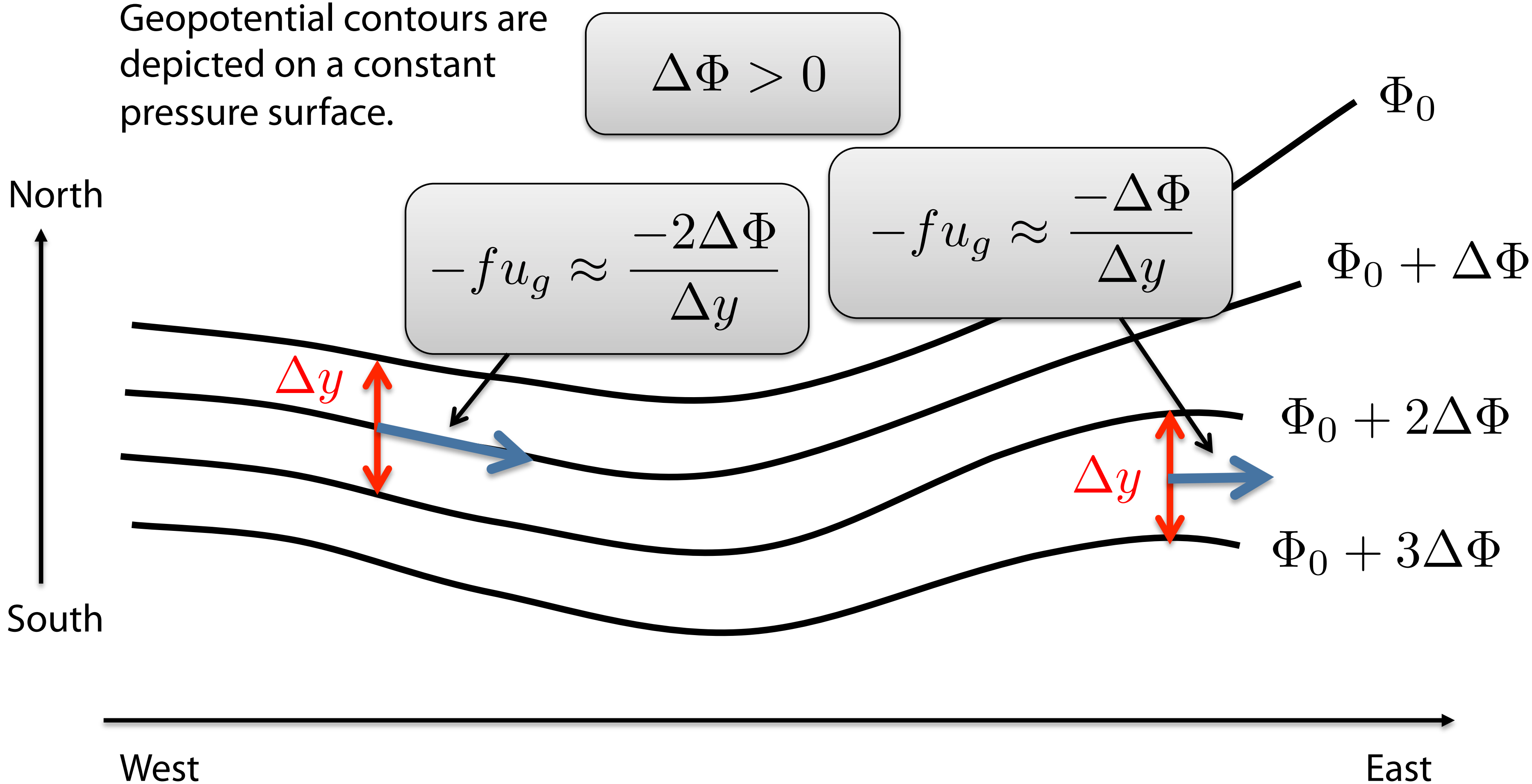
The Upper Troposphere

Geopotential contours are depicted on a constant pressure surface.



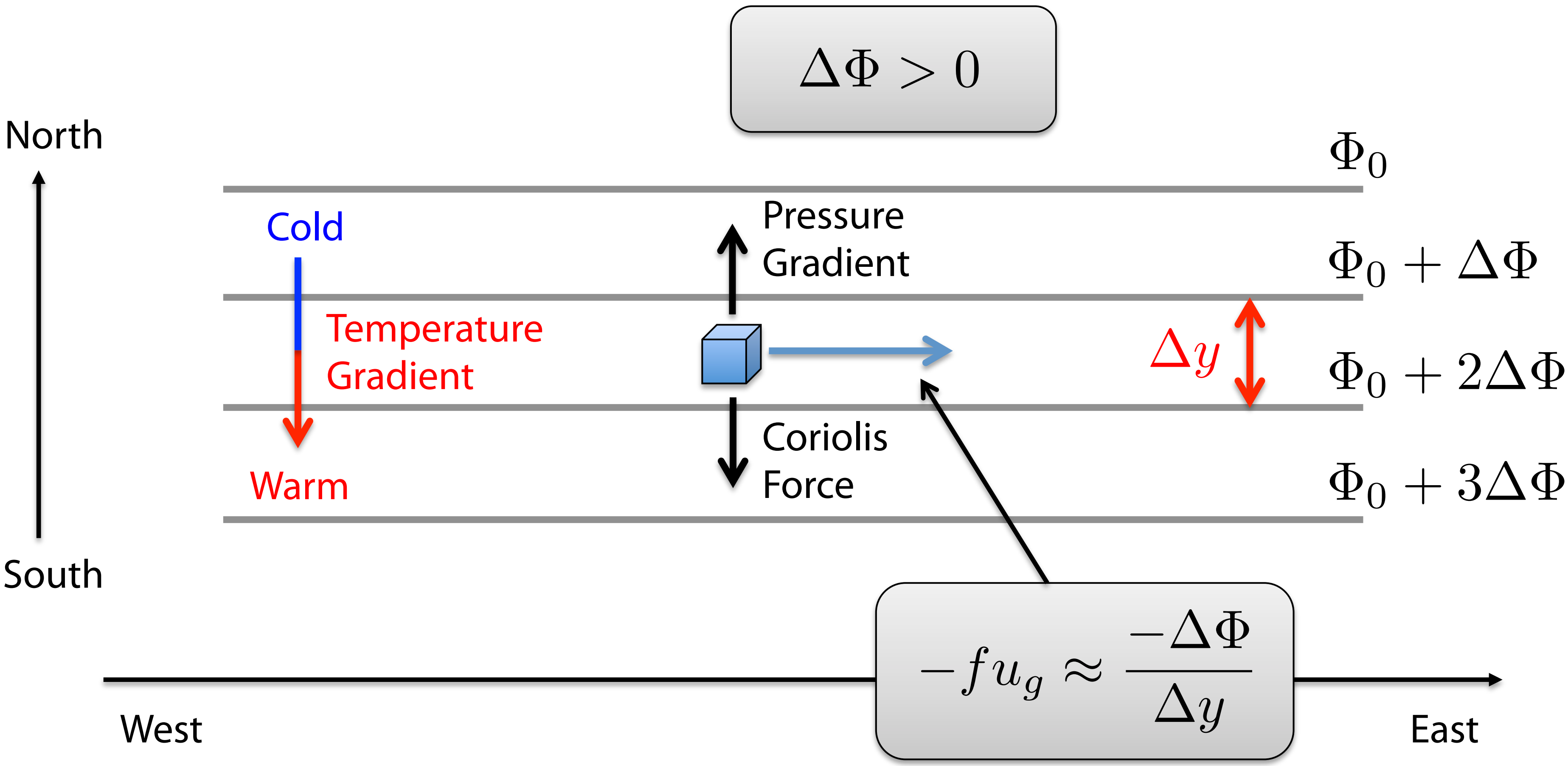
The Upper Troposphere

Geopotential contours are depicted on a constant pressure surface.



The Upper Troposphere

Think about this a minute



The Upper Troposphere

Think about this a minute

We have derived a formula for the **i** (eastward or x) component of the geostrophic wind.

We have estimated the derivatives based on *finite differences.* Recall we also used finite differences in deriving the equations of motion.

There is a consistency:

- Direction comes out correctly (towards east)
- The strength of the wind is proportional to the strength of the gradient.

The Upper Troposphere

Think about this a minute

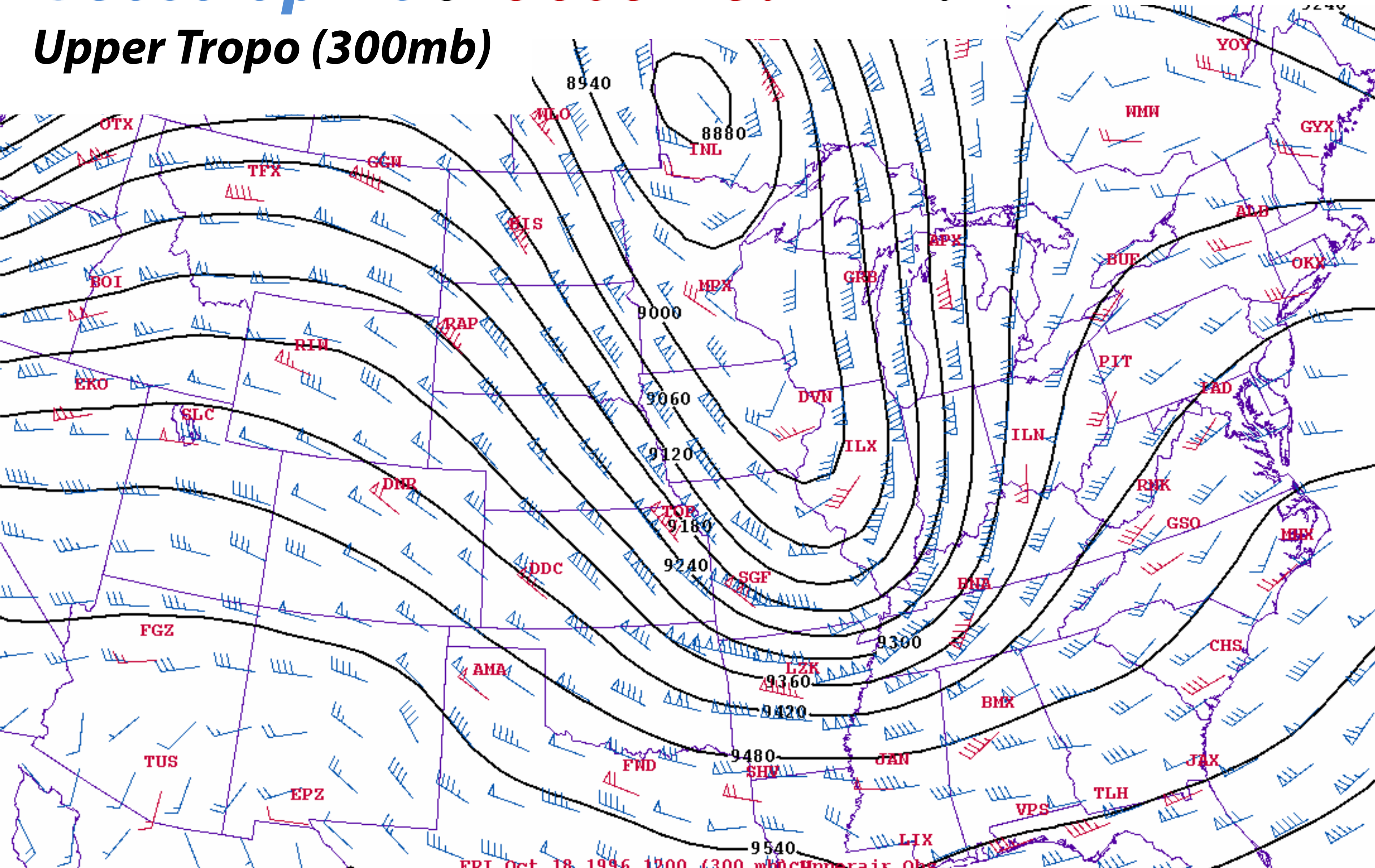
What about the observed wind?

- Flow is parallel to geopotential height lines
- ***But*** there is curvature in the flow as well.

IMPORTANT NOTE: This is not curvature due to the Earth, but curvature on a constant pressure surface due to bends and wiggles in the flow.

Geostrophic & *Observed* *Wind*

Upper Tropo (300mb)



The Upper Troposphere

What about the observed wind?

- Flow is parallel to geopotential height lines
- ***But*** there is curvature in the flow as well.

$$\left(\frac{\partial \Phi}{\partial x}\right)_p = f v_g$$

$$-\left(\frac{\partial \Phi}{\partial y}\right)_p = f u_g$$

Question: Where is curvature in these equations?

The Upper Troposphere

Think about the observed (upper level) wind:

- Flow is parallel to geopotential height lines
- There is curvature in the flow

Geostrophic balance describes flow parallel to geopotential height lines.

BUT Geostrophic balance does not account for curvature.

Question: How do we include curvature in our diagnostic equations?

Natural Coordinates

Question: Why do we need *another* coordinate system?

Our goal is to ***simplify*** the equations of motion. Sometimes complicated equations are simple if looked at in the right way.

At large scales, the atmosphere is in a state of balance. At large scales, mass fields (ρ, p, Φ) balance with wind fields (**u**).

But mass fields are generally much easier to observe than wind.

We need to describe balance between dominant terms: Pressure gradient, Coriolis and curvature of the flow.

Natural Coordinates

A "natural" set of direction vectors. When standing at a point, sometimes the only indication of direction is the direction of the flow.

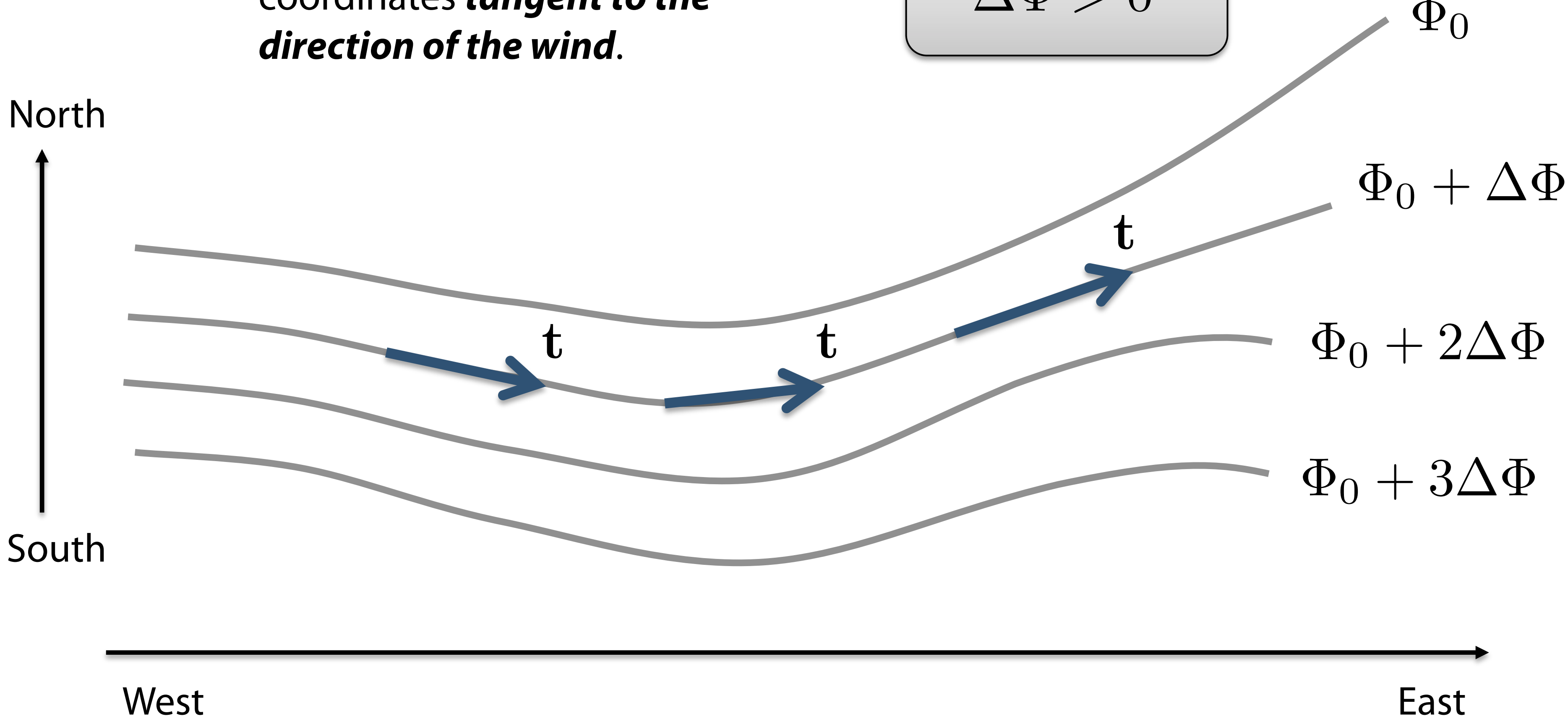
- Assumes no "local" changes in geopotential height. Flow is along contours of constant geopotential height.
- Assume horizontal flow only (on a constant pressure surface). An analogous method could be defined for height surfaces.
- Assume no friction (no viscous term)

⇨ Analogous to a Lagrangian parcel approach.

The Upper Troposphere

Define one component of these coordinates ***tangent to the direction of the wind***.

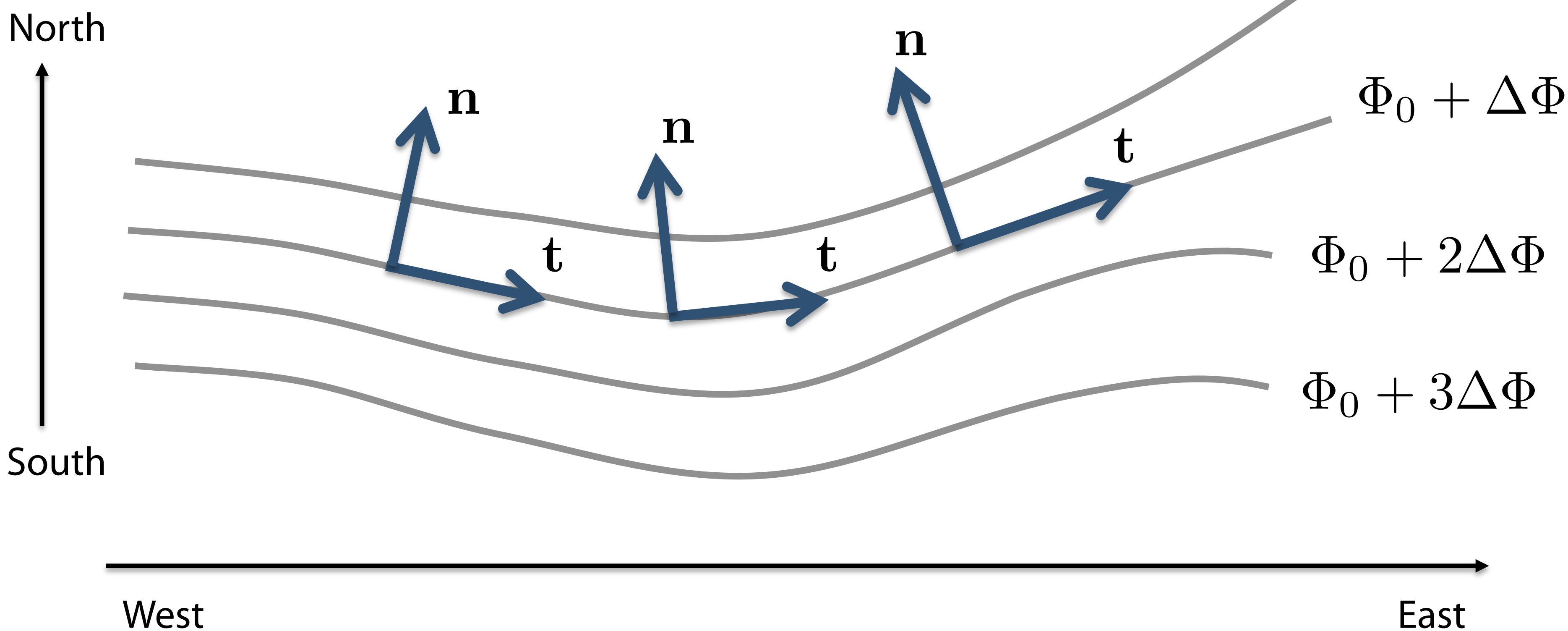
$$\Delta\Phi > 0$$



The Upper Troposphere

Define the other component of these coordinates ***normal to the direction of the wind***.

$$\Delta\Phi > 0$$



Natural Coordinates

Regardless of position:

- $\mathbf{t}$ always points in the direction of the flow
- $\mathbf{n}$ always points perpendicular to $\mathbf{t}$, to the left of the flow

$$\mathbf{n} = \mathbf{k} \times \mathbf{t}$$

Right-hand rule for vectors

Natural Coordinates

Advantages:

- We can look at a geopotential height (on a pressure surface) and estimate the winds.

    - In general it is difficult to measure winds, so we can now estimate winds from geopotential height (or pressure).

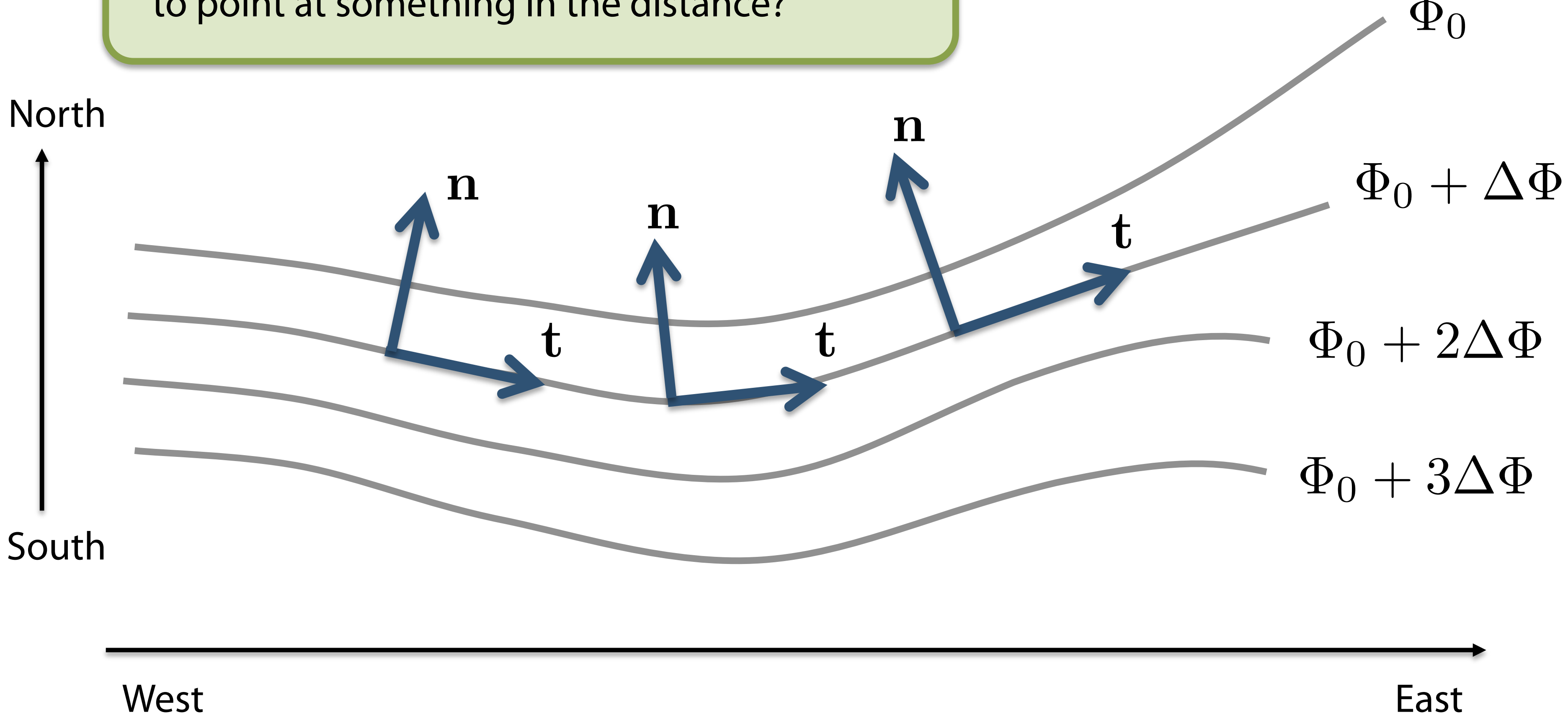
        - Useful for ***diagnostics*** and ***interpretation***.

Natural Coordinates

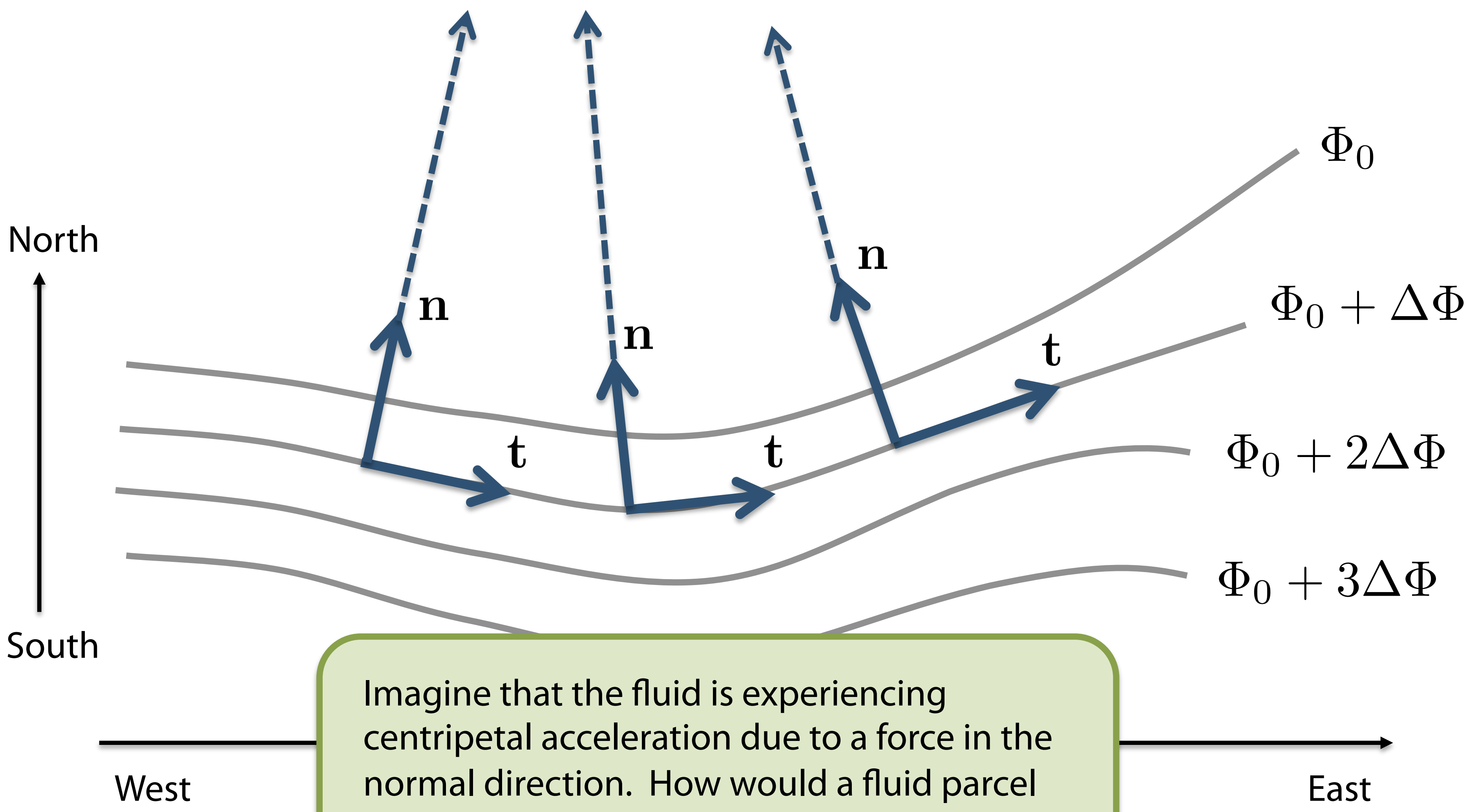
However, for diagnostics and interpretation of flows, we need an ***equation***.

Natural Coordinates

Do you observe that the normal arrows seem to point at something in the distance?

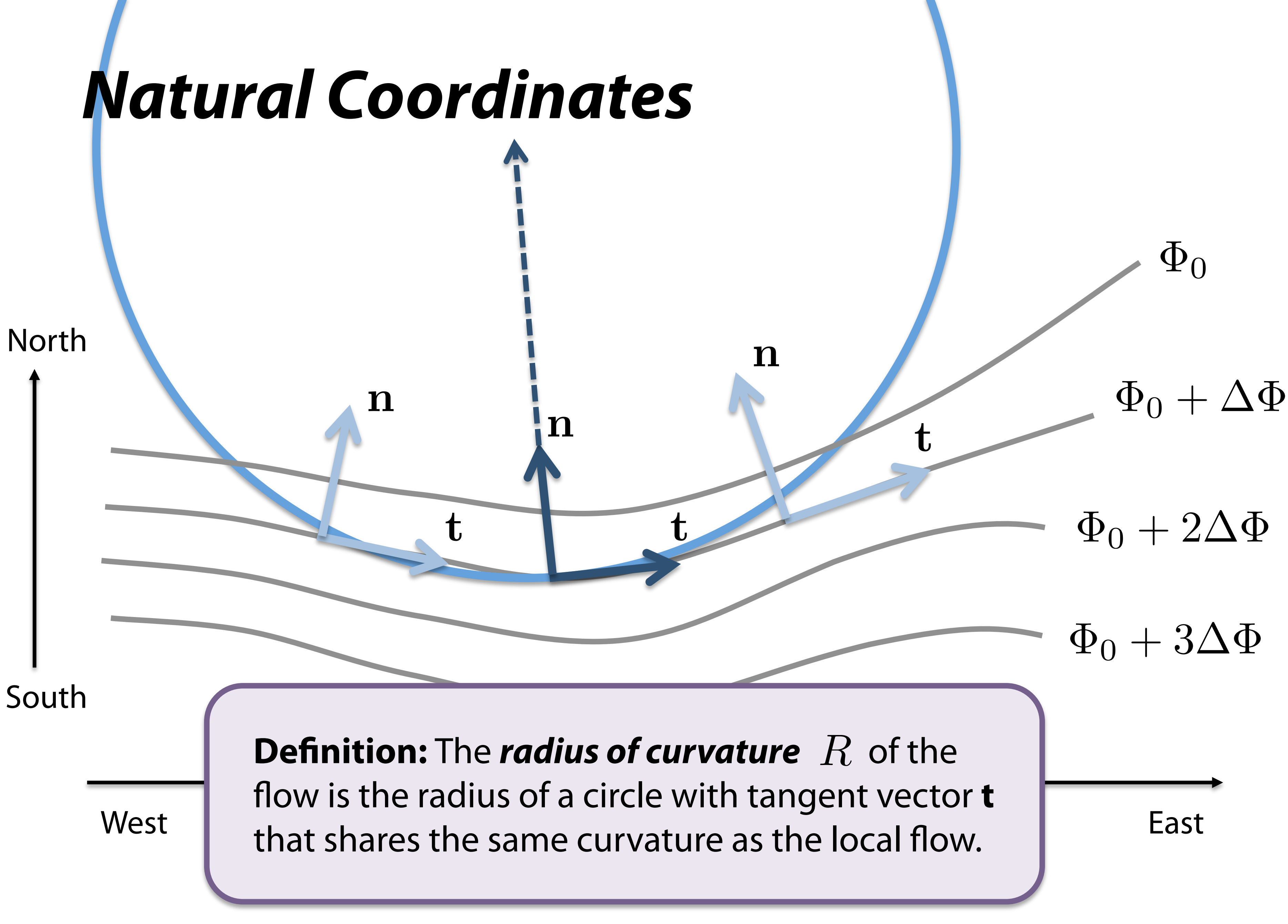


Natural Coordinates



Imagine that the fluid is experiencing centripetal acceleration due to a force in the normal direction. How would a fluid parcel react?

Natural Coordinates



Definition: The ***radius of curvature*** R of the flow is the radius of a circle with tangent vector **t** that shares the same curvature as the local flow.

Natural Coordinates

Velocity in Natural Coordinates

$$\mathbf{u} = V\mathbf{t} \Rightarrow V = |\mathbf{u}|$$

Velocity Vector

Velocity Magnitude

Unit vector tangent to the flow

Simplifications:

1. Velocity is **always** in the direction of **t**
2. The value of u is **always positive**

Natural Coordinates

Acceleration in Natural Coordinates

$$\frac{D\mathbf{u}}{Dt} = \frac{D(V\mathbf{t})}{Dt} = \frac{DV}{Dt}\mathbf{t} + V\frac{D\mathbf{t}}{Dt}$$

Definition of acceleration

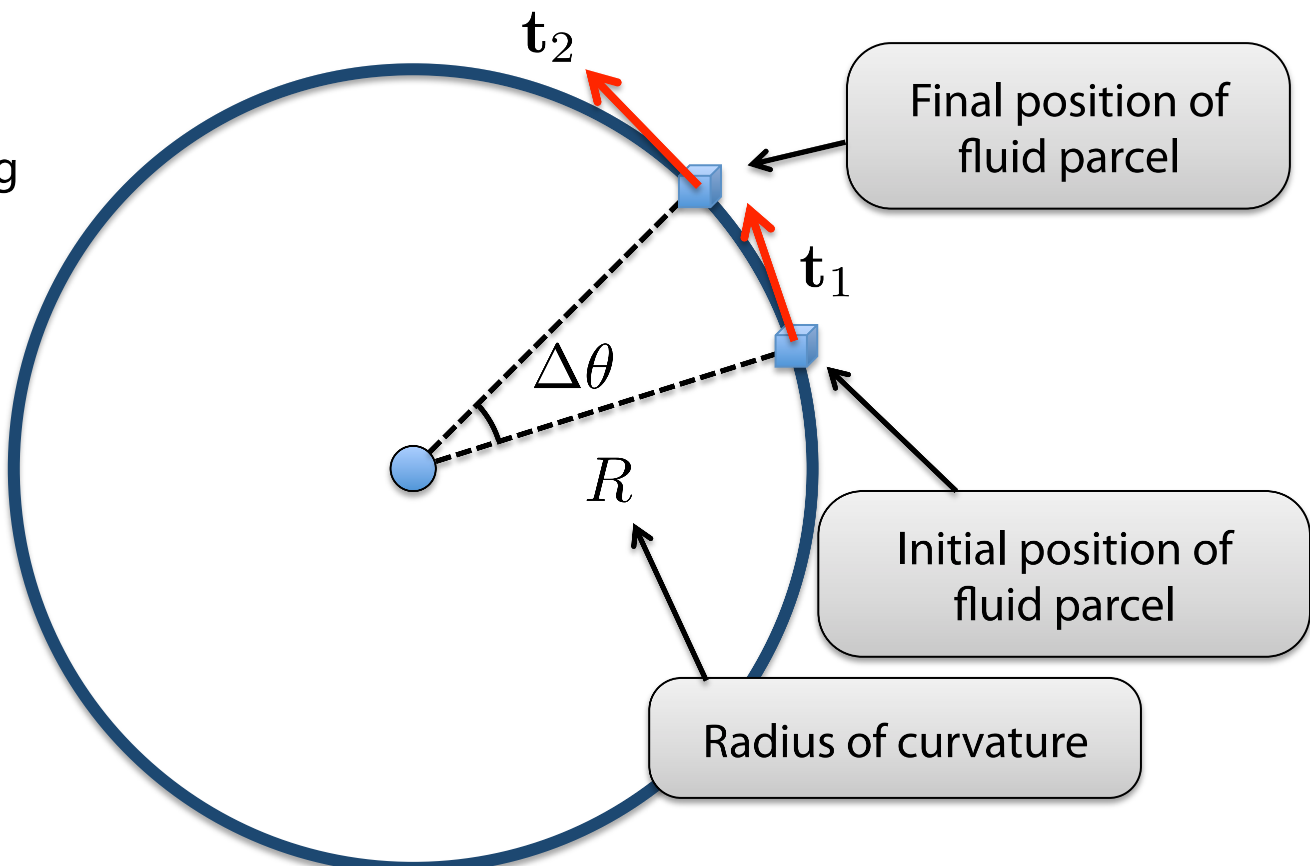
Change in speed

Change in direction

Natural Coordinates

Question: How do we get $\dfrac{D\mathbf{t}}{Dt}$ as a function of V, R ?

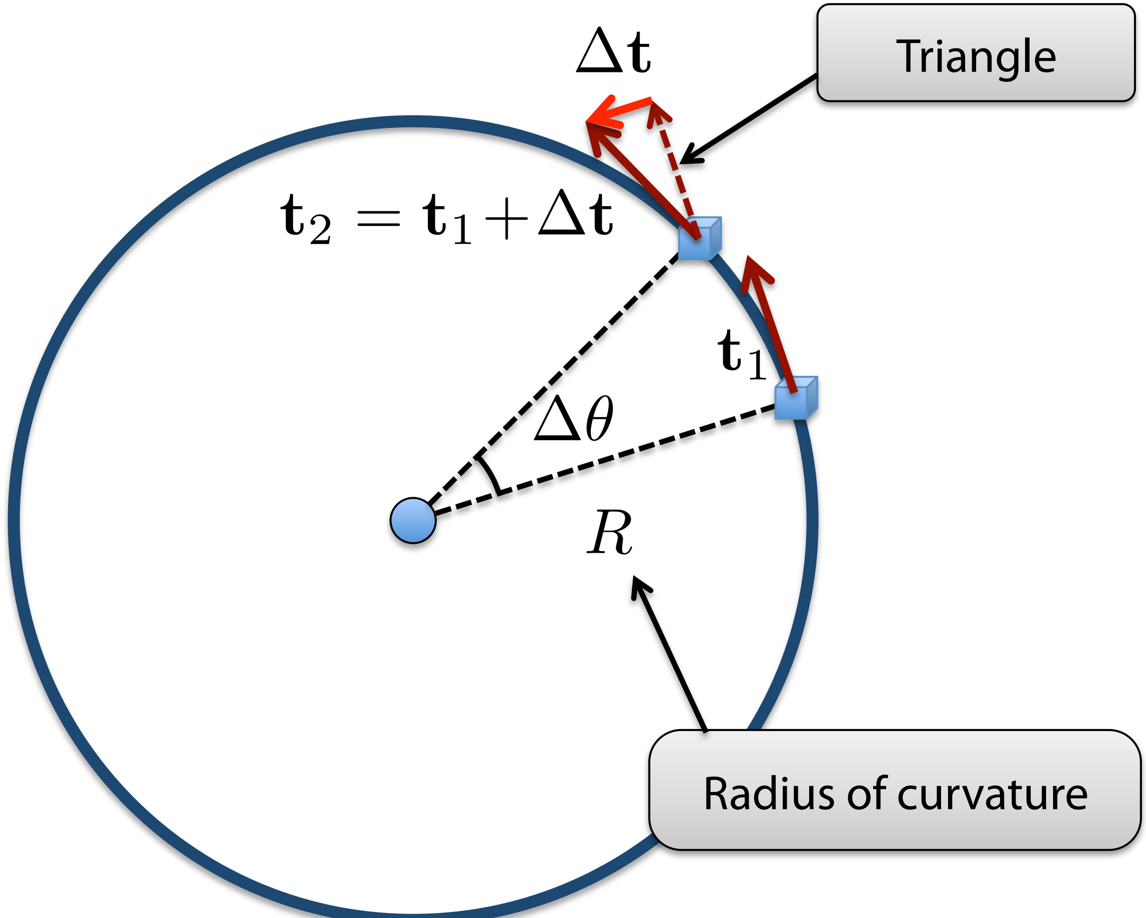
For simplicity, consider a fluid parcel moving along a circular trajectory.



Recall the use of circle geometry (from derivation of Coriolis / centrifugal force)

Natural Coordinates

Between the initial and final positions, the tangent vector changes by an amount $\Delta\mathbf{t}$.



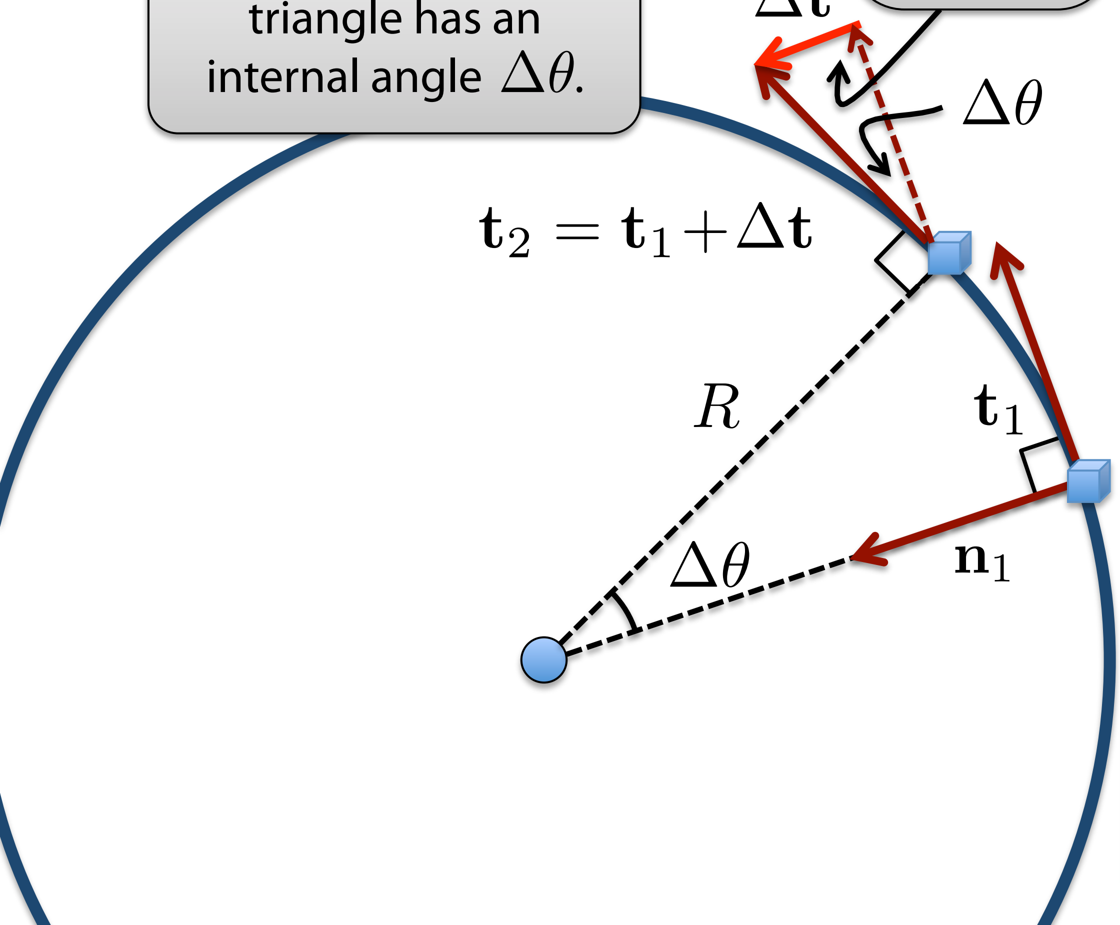
Recall the use of circle geometry (from derivation of Coriolis / centrifugal force)

Natural Coordinates

Zoomed in…

Using geometry, this triangle has an internal angle $\Delta\theta$.

Define angle α



Use the law of sines and the fact that tangent vectors have unit length:

$$\sin\alpha = \sin\left(\pi - \alpha - \Delta\theta\right)$$

Since all angles are < 90°

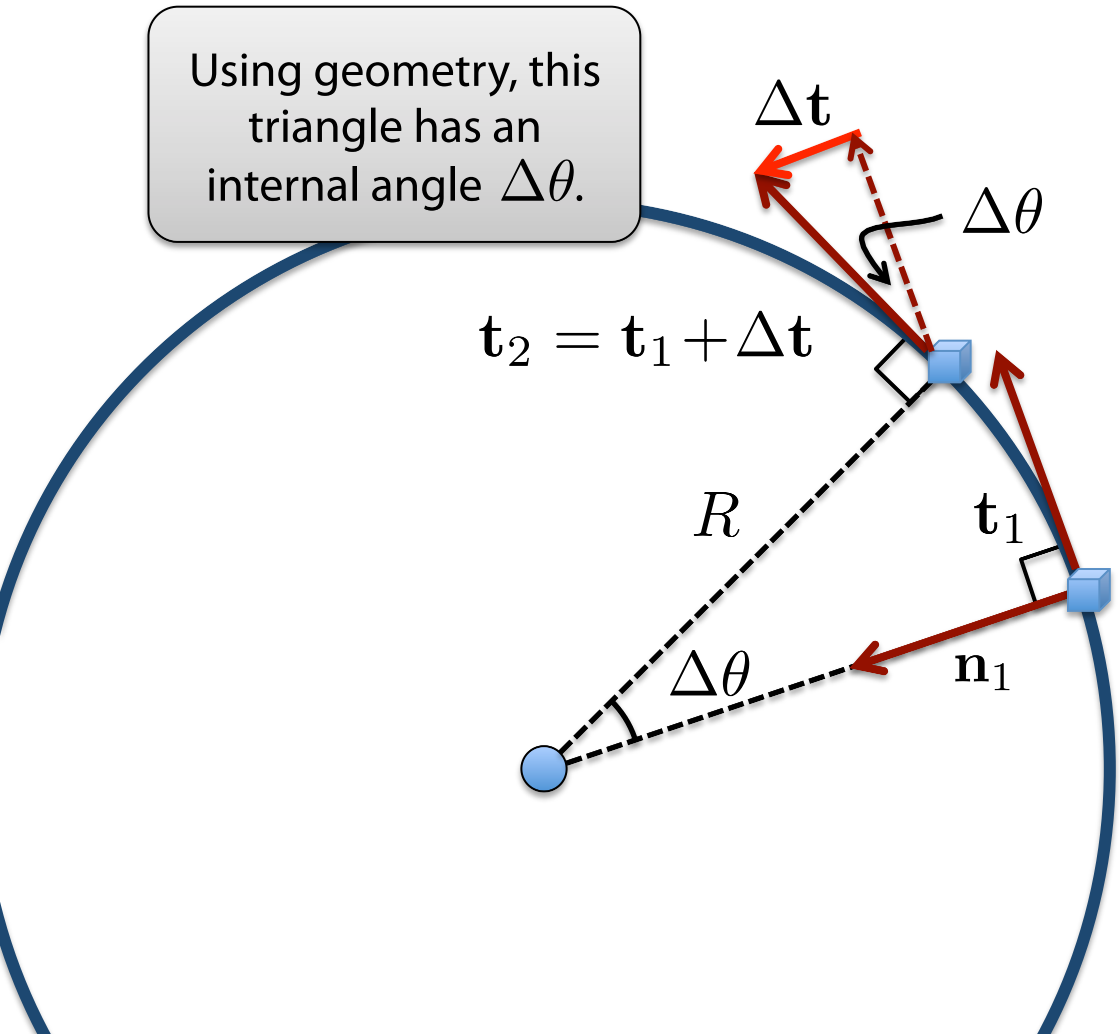
$$\alpha = \pi - \alpha - \Delta\theta$$

$$\alpha = \frac{\pi}{2} - \frac{\Delta\theta}{2}$$

For small displacements, $\Delta\mathbf{t}$ will point in the same direction as $\mathbf{n}_1$ (= 90° to $\mathbf{t}_1$)

Natural Coordinates

Zoomed in…



Observe that for small displacements (and using the fact that tangent vectors are unit length):

$$|\Delta\mathbf{t}| \approx \Delta\theta$$

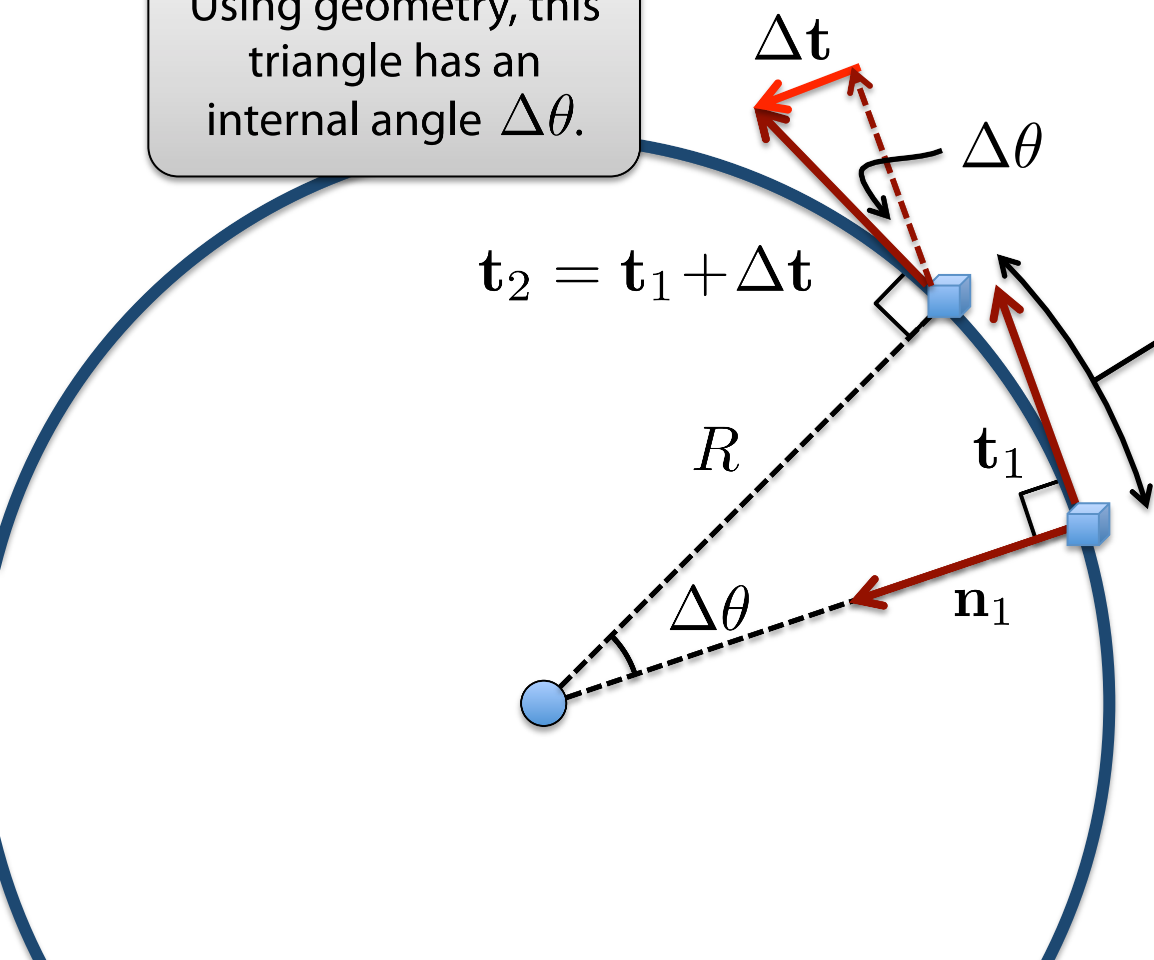
Consequently:

$$\Delta\mathbf{t} \approx \Delta\theta\mathbf{n}_1$$

Natural Coordinates

Zoomed in…

Using geometry, this triangle has an internal angle $\Delta\theta$.



From the last slide:

$$\Delta\mathbf{t} \approx \Delta\theta\mathbf{n}_1$$

Distance traveled by fluid parcel

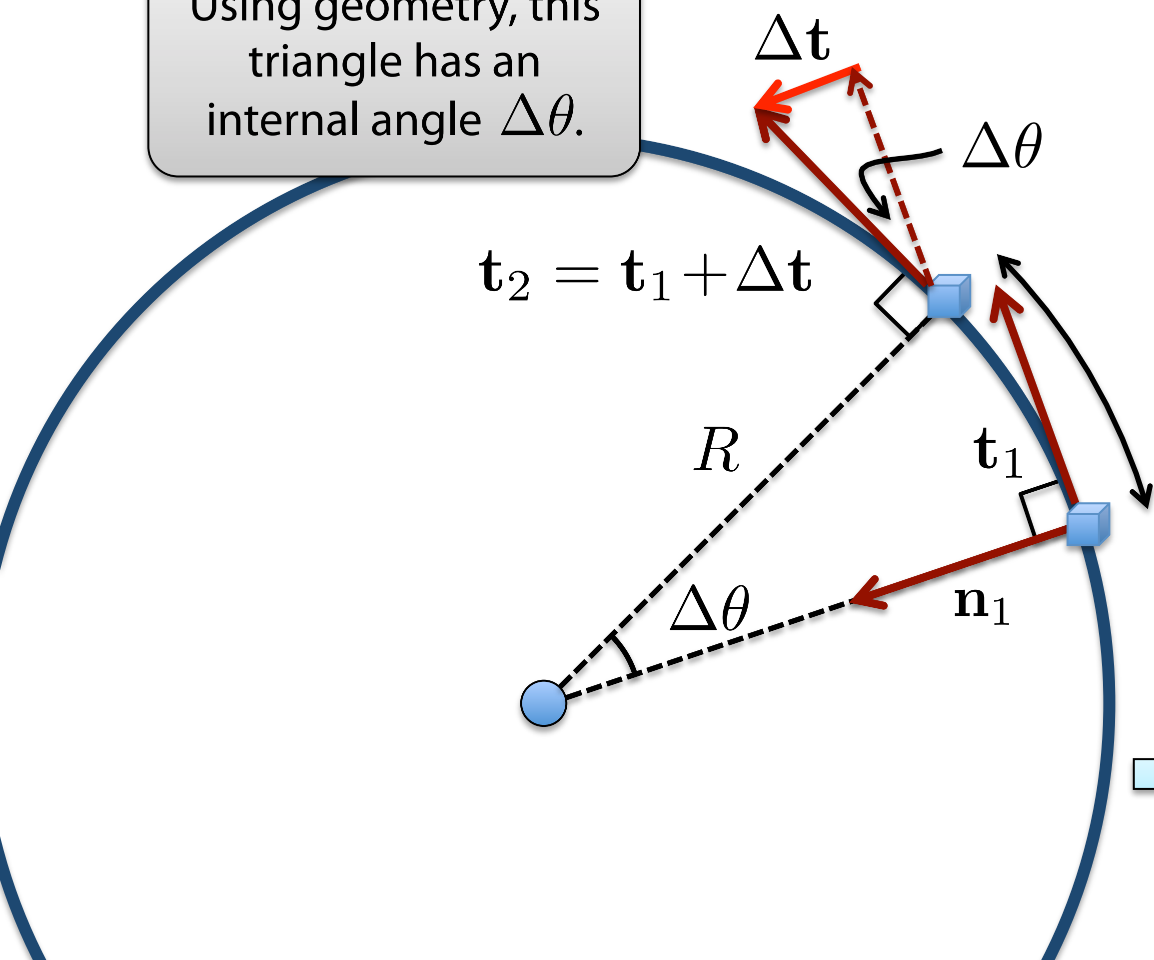
$$\Delta s = R\Delta\theta$$

$$\Rightarrow \quad \Delta\mathbf{t} \approx \frac{\Delta s}{R}\mathbf{n}_1$$

Natural Coordinates

Zoomed in…

Using geometry, this triangle has an internal angle $\Delta\theta$.



From the last slide:

$$\frac{\Delta\mathbf{t}}{\Delta t} \approx \frac{1}{R}\frac{\Delta s}{\Delta t}\mathbf{n}_1$$

Distance / Time = Velocity

In the limit of $\Delta t \to 0$

$$\boxed{\frac{D\mathbf{t}}{Dt} = \frac{1}{R}\frac{Ds}{Dt}\mathbf{n}_1 = \frac{V}{R}\mathbf{n}}$$

Natural Coordinates

Remember our goal is to quantify acceleration…

$$\frac{D\mathbf{u}}{Dt} = \frac{D(V\mathbf{t})}{Dt} = \frac{DV}{Dt}\mathbf{t} + V\frac{D\mathbf{t}}{Dt}$$

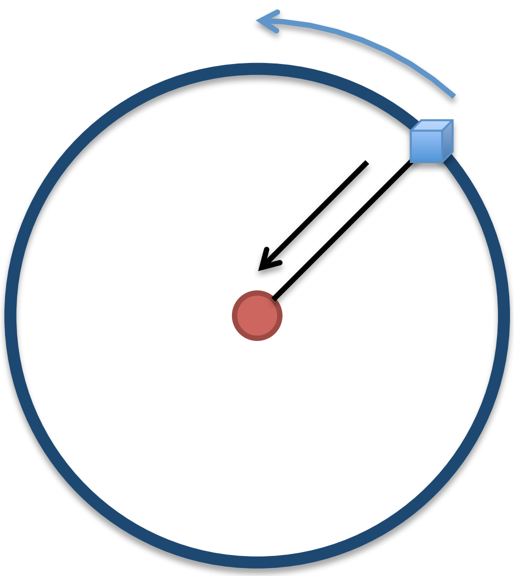
$$\boxed{\frac{D\mathbf{t}}{Dt} = \frac{V}{R}\mathbf{n}}$$

$$\frac{D\mathbf{u}}{Dt} = \frac{DV}{Dt}\mathbf{t} + \frac{V^2}{R}\mathbf{n}$$

Change in speed

?

Natural Coordinates



Recall from physics 101 centripetal acceleration:

An object traveling at velocity V forced to remain along a circular trajectory will experience a centripetal force with magnitude V^2/R towards the center of the circle

$$\frac{D\mathbf{u}}{Dt} = \frac{DV}{Dt}\mathbf{t} + \frac{V^2}{R}\mathbf{n}$$

Change in speed

Centripetal acceleration due to curvature in the flow

Momentum Equation

Now that we have an equation for change in horizontal momentum in terms of tangental and normal vectors, we would like to derive a momentum equation.

The momentum equation must contain terms:

- Acceleration
- Coriolis force
- Pressure gradient force

Momentum Equation

Coriolis Force

Coriolis force always acts normal to the velocity, with magnitude f:

$$\boxed{\mathbf{F}_{cor} = -f\mathbf{k} \times \mathbf{u} = -fV\mathbf{n}}$$

Momentum Equation

Pressure Gradient Force

Pressure gradient force acts in the opposing direction of the pressure gradient. On a surface of constant pressure this leads to:

$$\boxed{\mathbf{F}_p = -\nabla_p \Phi = -\left(\mathbf{t}\frac{\partial \Phi}{\partial s} + \mathbf{n}\frac{\partial \Phi}{\partial n}\right)}$$

Momentum Equation

Using the vector form of the momentum equation:

$$\frac{D\mathbf{u}}{Dt} + f\mathbf{k} \times \mathbf{u} = -\nabla_p \Phi$$

Make all substitutions:

$$\mathbf{F}_p = -\nabla_p \Phi = -\left(\mathbf{t} \frac{\partial \Phi}{\partial s} + \mathbf{n} \frac{\partial \Phi}{\partial n} \right)$$

$$\mathbf{F}_{cor} = -f\mathbf{k} \times \mathbf{u} = -fV\mathbf{n}$$

$$\boxed{\frac{DV}{Dt}\mathbf{t} + \frac{V^2}{R}\mathbf{n} + fV\mathbf{n} = -\left(\mathbf{t} \frac{\partial \Phi}{\partial s} + \mathbf{n} \frac{\partial \Phi}{\partial n} \right)}$$

Momentum Equation

$$\frac{DV}{Dt}\mathbf{t} + \frac{V^2}{R}\mathbf{n} + fV\mathbf{n} = -\left(\mathbf{t}\frac{\partial \Phi}{\partial s} + \mathbf{n}\frac{\partial \Phi}{\partial n}\right)$$

In component form:

$$\frac{DV}{Dt} = -\frac{\partial \Phi}{\partial s}$$

$$\frac{V^2}{R} + fV = -\frac{\partial \Phi}{\partial n}$$

Along flow direction ($\mathbf{t}$)

Across flow direction ($\mathbf{n}$)

Momentum Equation

Is this a simplification?

Recall we are only considering flow along geopotential height contours:

$$\cancelto{0}{\frac{DV}{Dt}} = -\cancelto{0}{\frac{\partial \Phi}{\partial s}}$$

Along flow direction ($\mathbf{t}$)

$$\frac{V^2}{R} + fV = -\frac{\partial \Phi}{\partial n}$$

Across flow direction ($\mathbf{n}$)

By using natural coordinates, we only require one diagnostic equation to describe velocity.

Momentum Equation

One diagnostic equation for curved flow:

$$\frac{V^2}{R} + fV = -\frac{\partial \Phi}{\partial n}$$

Centripetal acceleration

Coriolis force

Pressure gradient force

Question: How does this generalize the geostrophic approximation?