Introduction to Atmospheric Dynamics
Chapter 2

Paul A. Ullrich
paullrich@ucdavis.edu

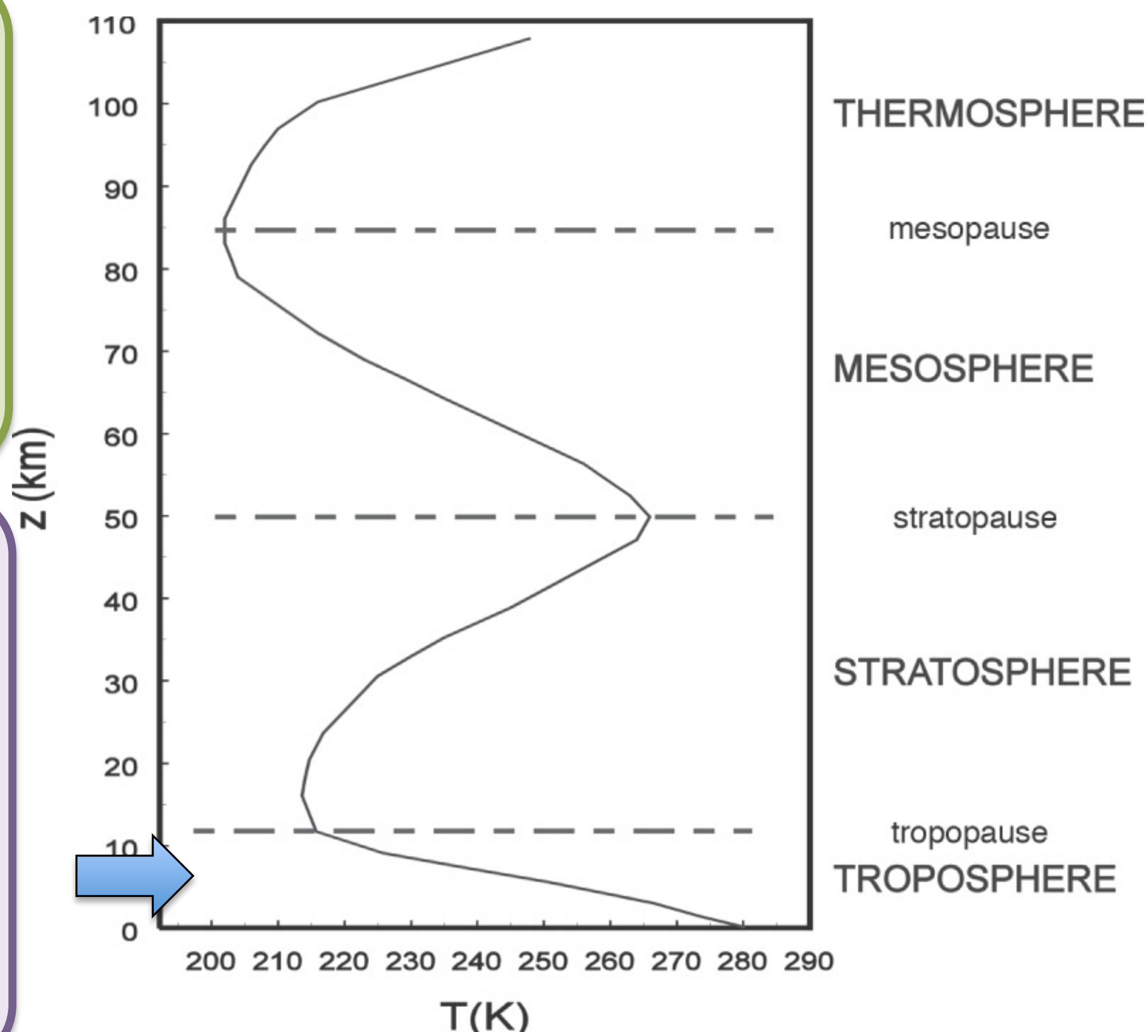
Part 3: Buoyancy and Convection

Vertical Structure

This cooling with height is related to the dynamics of the atmosphere. The change of temperature with height is called the **lapse rate**.

Definition: The **lapse rate** is defined as the rate (for instance in K/km) at which temperature decreases with height.

$$\Gamma \equiv -\frac{\partial T}{\partial z}$$



Lapse Rate

For a dry adiabatic, stable, hydrostatic atmosphere the potential temperature θ does not vary in the vertical direction:

$$\frac{\partial \theta}{\partial z} = 0$$

In a dry adiabatic, hydrostatic atmosphere the temperature T must decrease with height. How quickly does the temperature decrease?

Note: Use $\theta = T\left(\frac{p_0}{p}\right)^{R/c_p}$

Lapse Rate

The adiabatic change in temperature with height is

$$\frac{T}{\theta}\frac{\partial\theta}{\partial z} = \frac{\partial T}{\partial z} + \frac{g}{c_p}$$

For dry adiabatic, hydrostatic atmosphere:

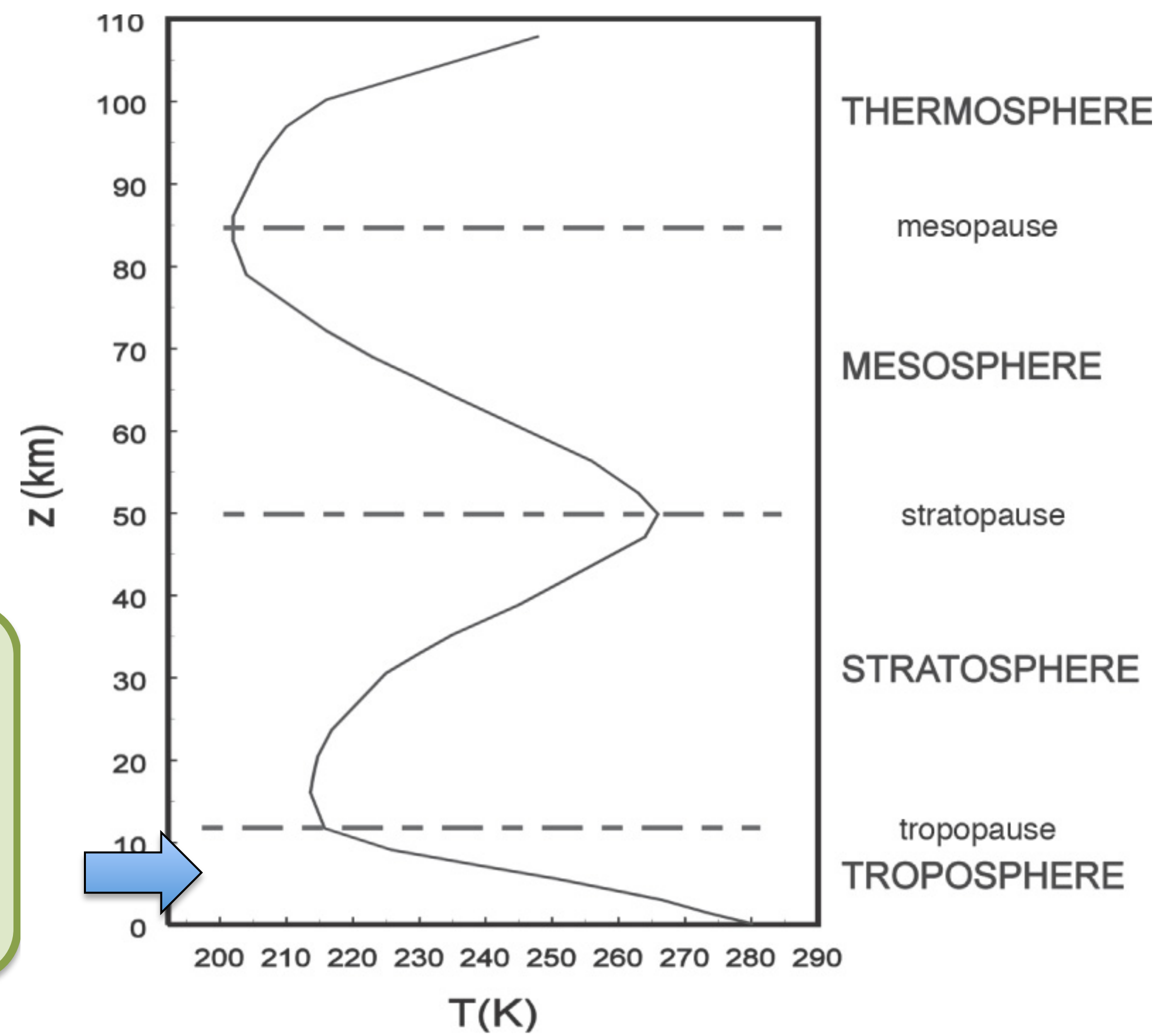
$$-\frac{\partial T}{\partial z} = \frac{g}{c_p} \equiv \Gamma_d$$

Definition: The **dry adiabatic lapse rate** is defined as the rate (for instance in K/km) at which the temperature of an air parcel will decrease with height if raised adiabatically.

$$\Gamma_d \equiv \frac{g}{c_p}$$

$$\frac{g}{c_p} \approx 9.8\ \mathrm{K\ km}^{-1}$$

Lapse Rate



This profile should be very close to the adiabatic lapse rate in a dry atmosphere.

Fundamentals

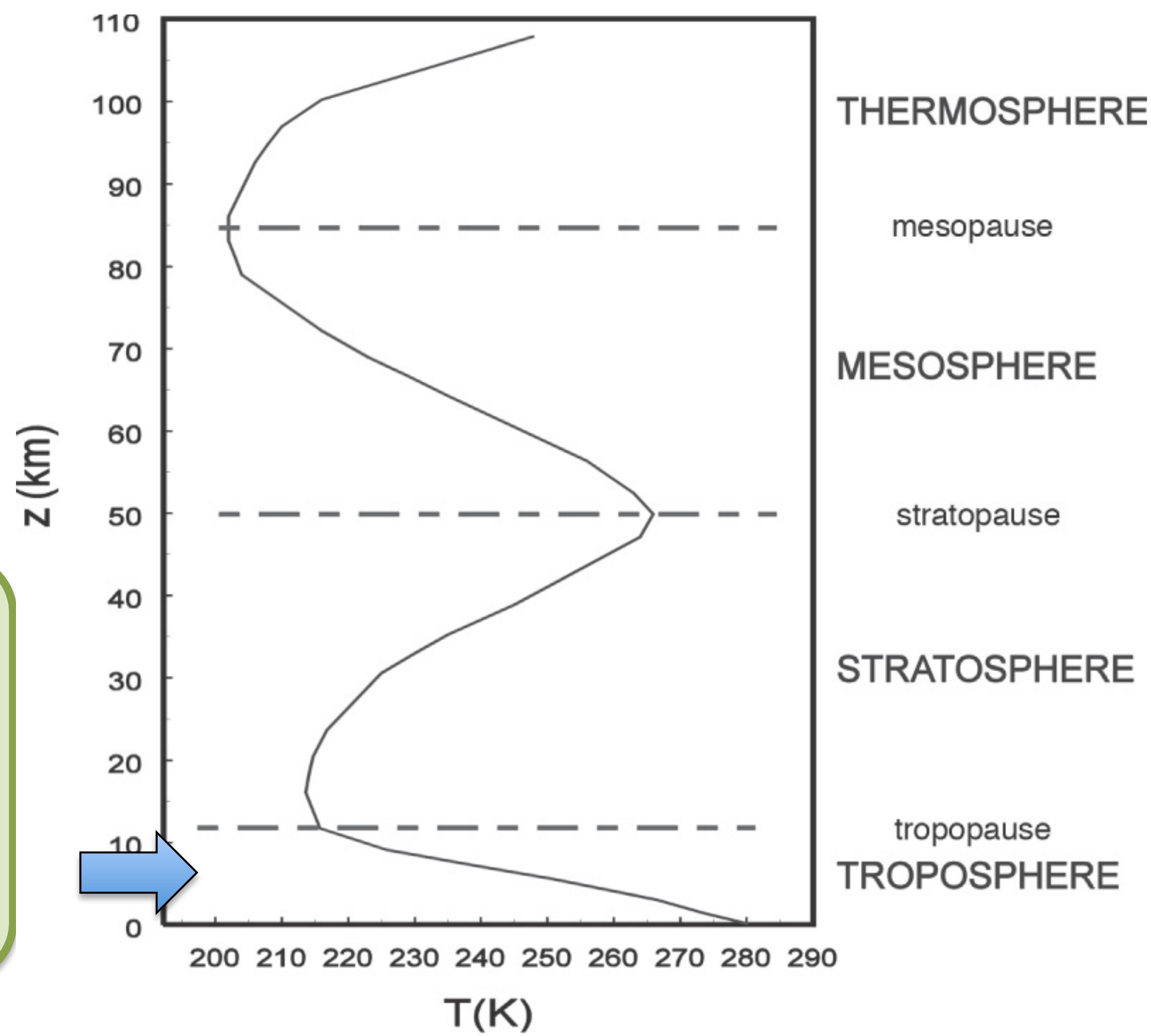
Even in adiabatic motion, with no external source of heating, if a parcel moves up or down its temperature will change.

- What if a parcel moves about a surface of constant pressure?
- What if a parcel moves about a surface of constant height?

If the atmosphere is in adiabatic balance, the temperature still changes with height.

Adiabatic does not mean isothermal. It means there is no external heating or cooling.

Lapse Rate



In fact, the temperature decline here is not the dry adiabatic lapse rate, because the atmosphere (in general) is not dry.

Static Stability and Moisture

In general, the atmosphere is not dry.

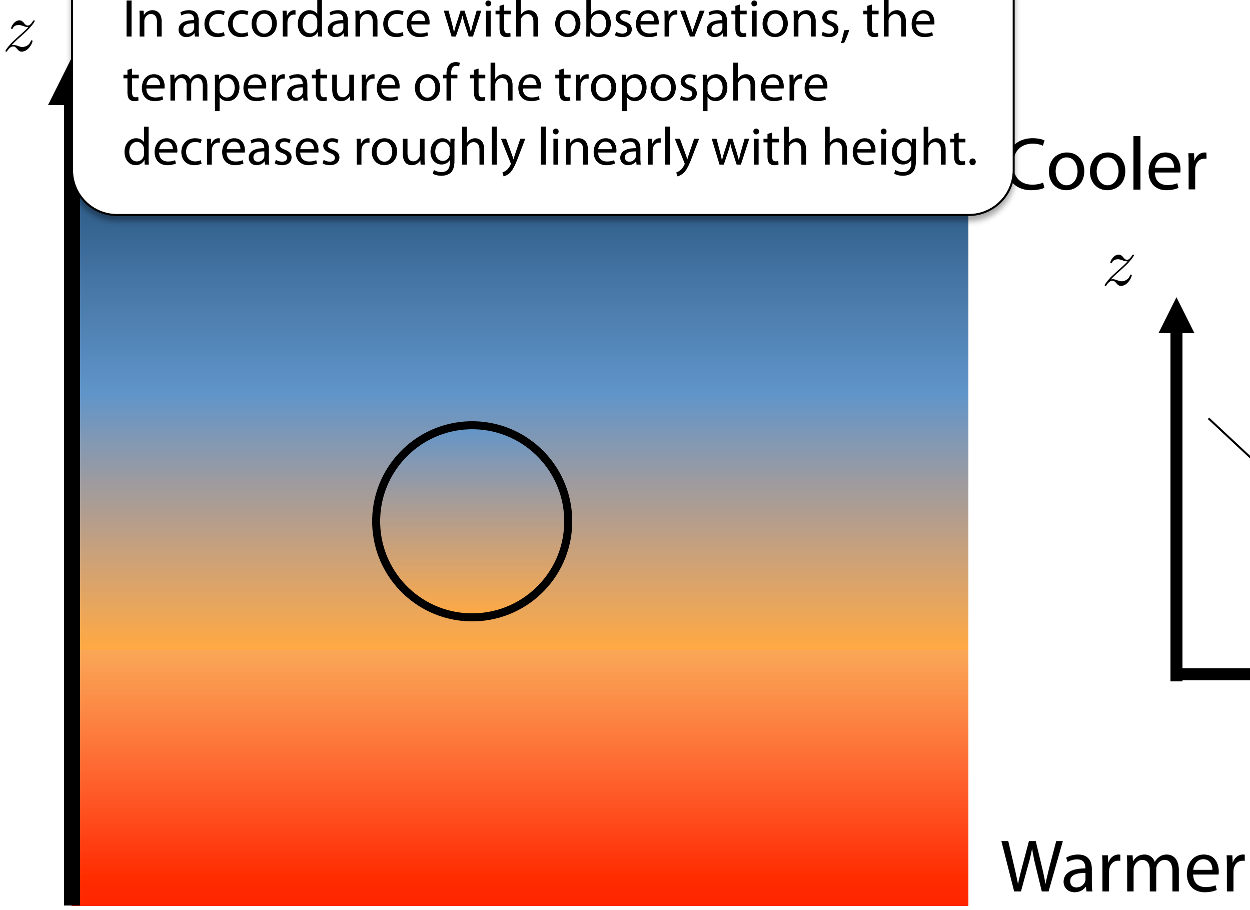
If air reaches saturation (and the conditions are right for cloud formation), vapor will condense to liquid or solid and release energy $(J \neq 0)$

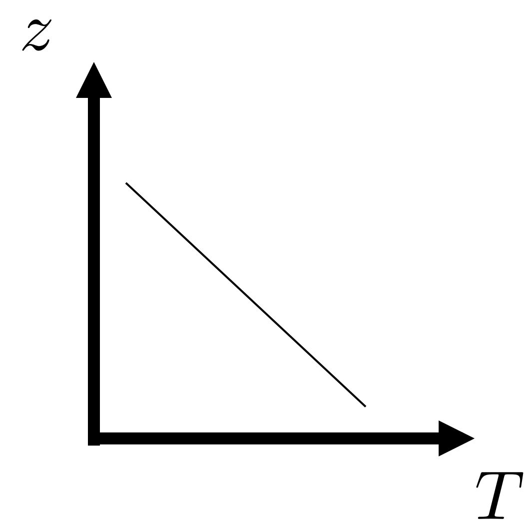
Moist (saturated) adiabatic lapse rate: $\sim 5 \ \mathrm{K/km}$

Average lapse rate in the troposphere: $\sim 6.5 \ \mathrm{K/km}$

Dry Convection

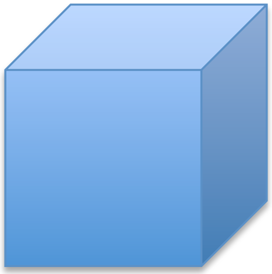
In accordance with observations, the temperature of the troposphere decreases roughly linearly with height.





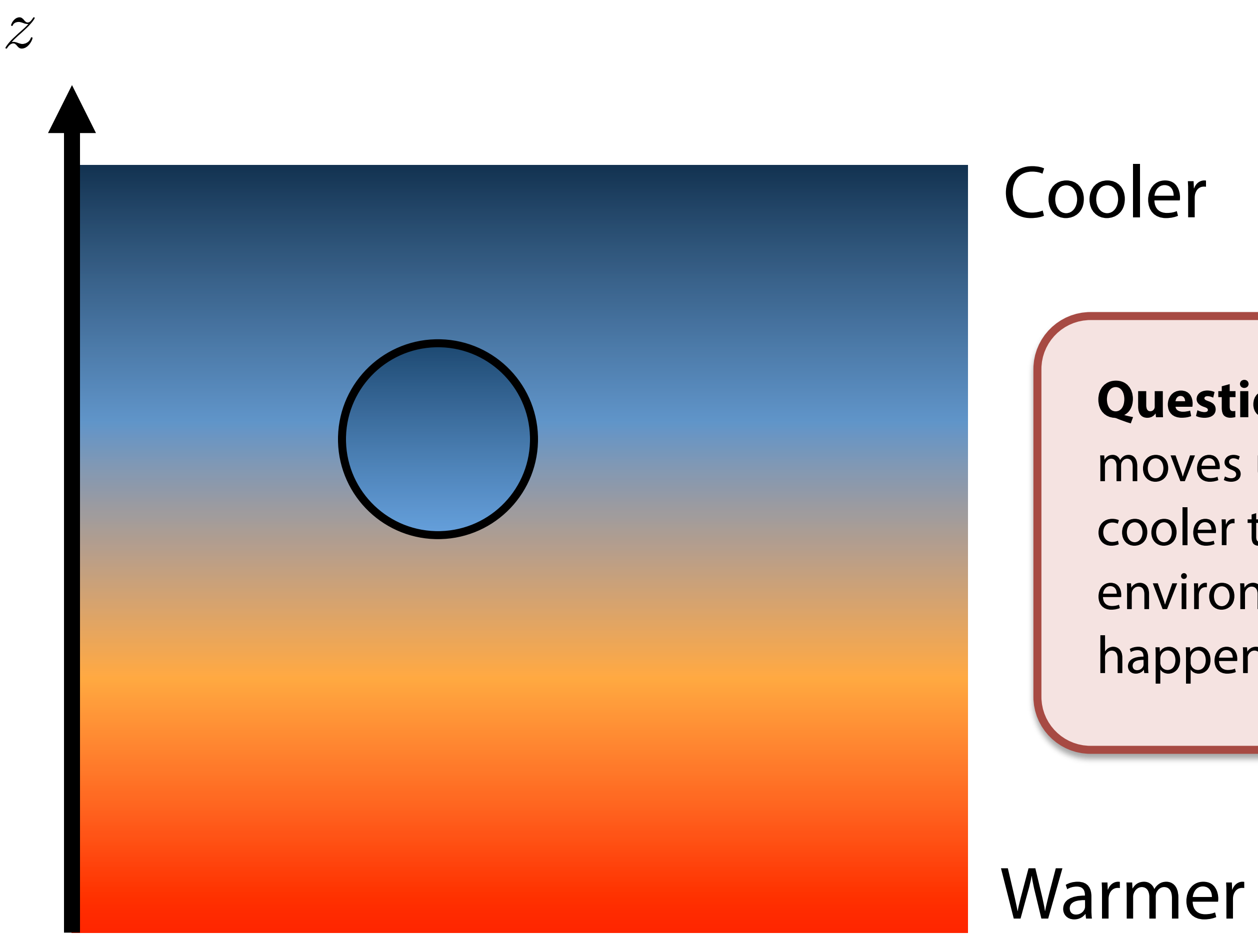
Dry Convection

The Parcel Method



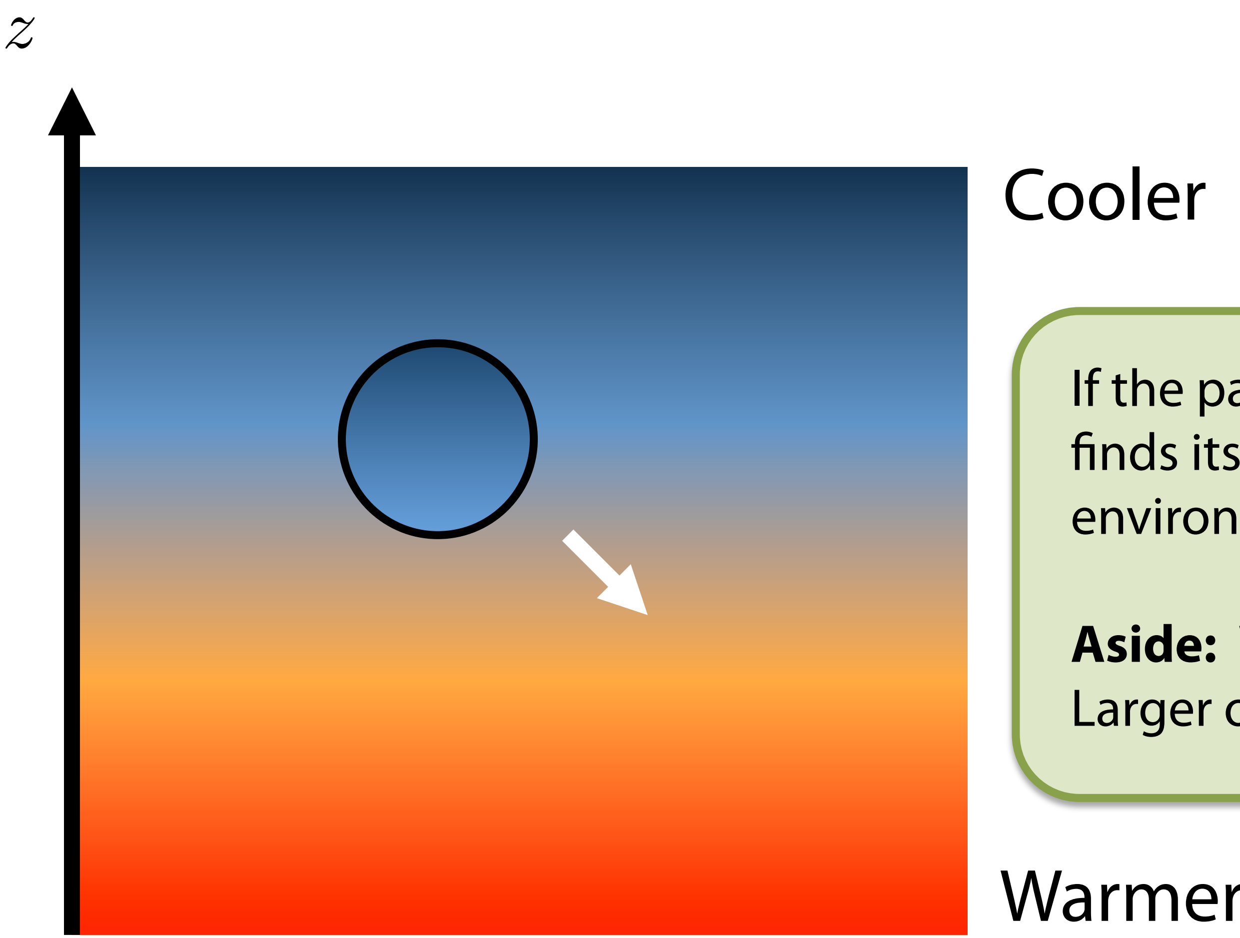
- Consider an arbitrary air parcel sitting in the atmosphere that is displaced up or down.
- Assume that the pressure adjusts instantaneously; that is, the parcel assumes the pressure of the altitude to which it is displaced.
- As the parcel moves its temperature will change according to the adiabatic lapse rate. That is, the motion is without the addition or subtraction of energy ($J = 0$ in thermodynamic equation)

Dry Convection



Question: If the parcel moves up and finds itself cooler than the environment, what will happen?

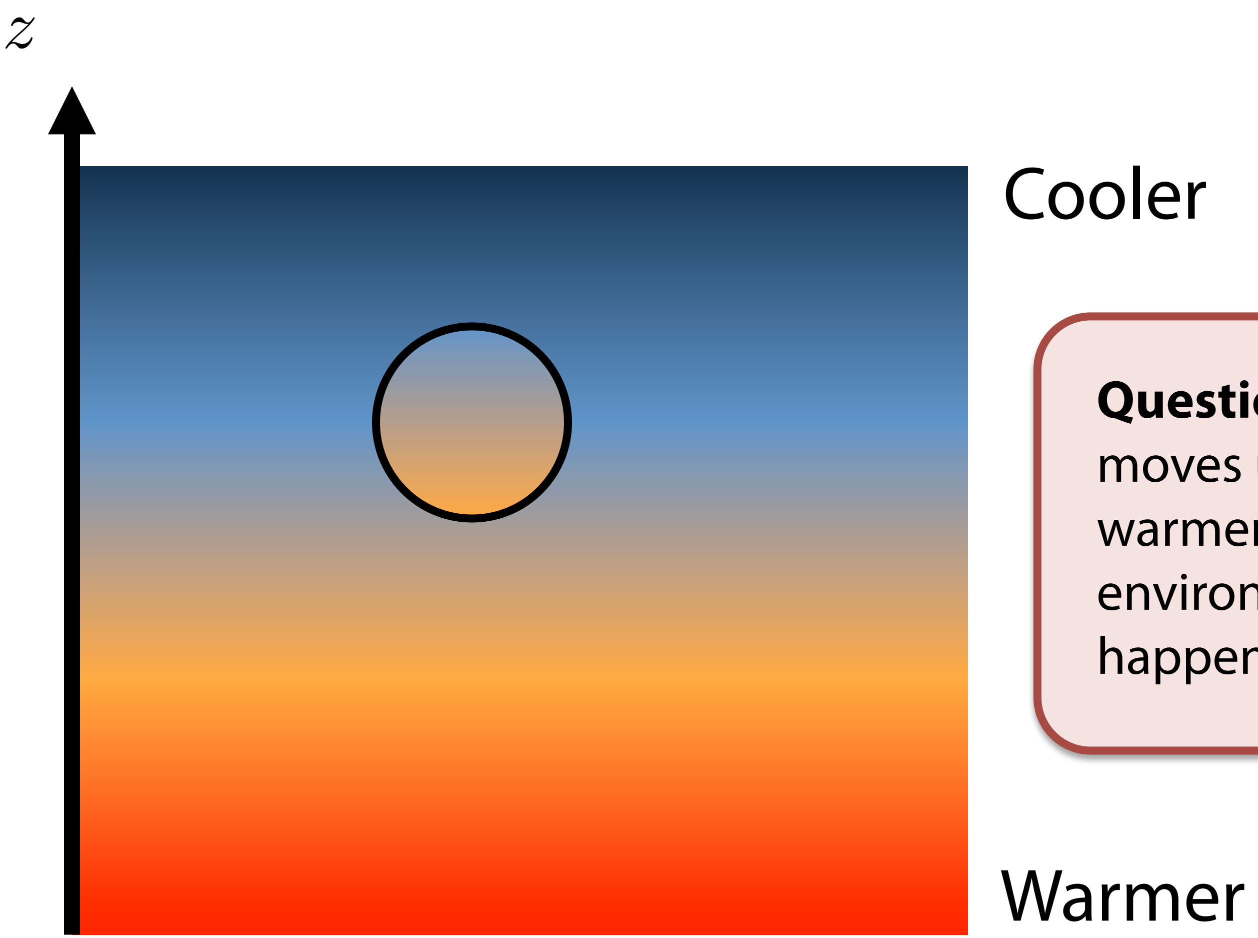
Dry Convection



If the parcel moves up and finds itself cooler than the environment then it will sink.

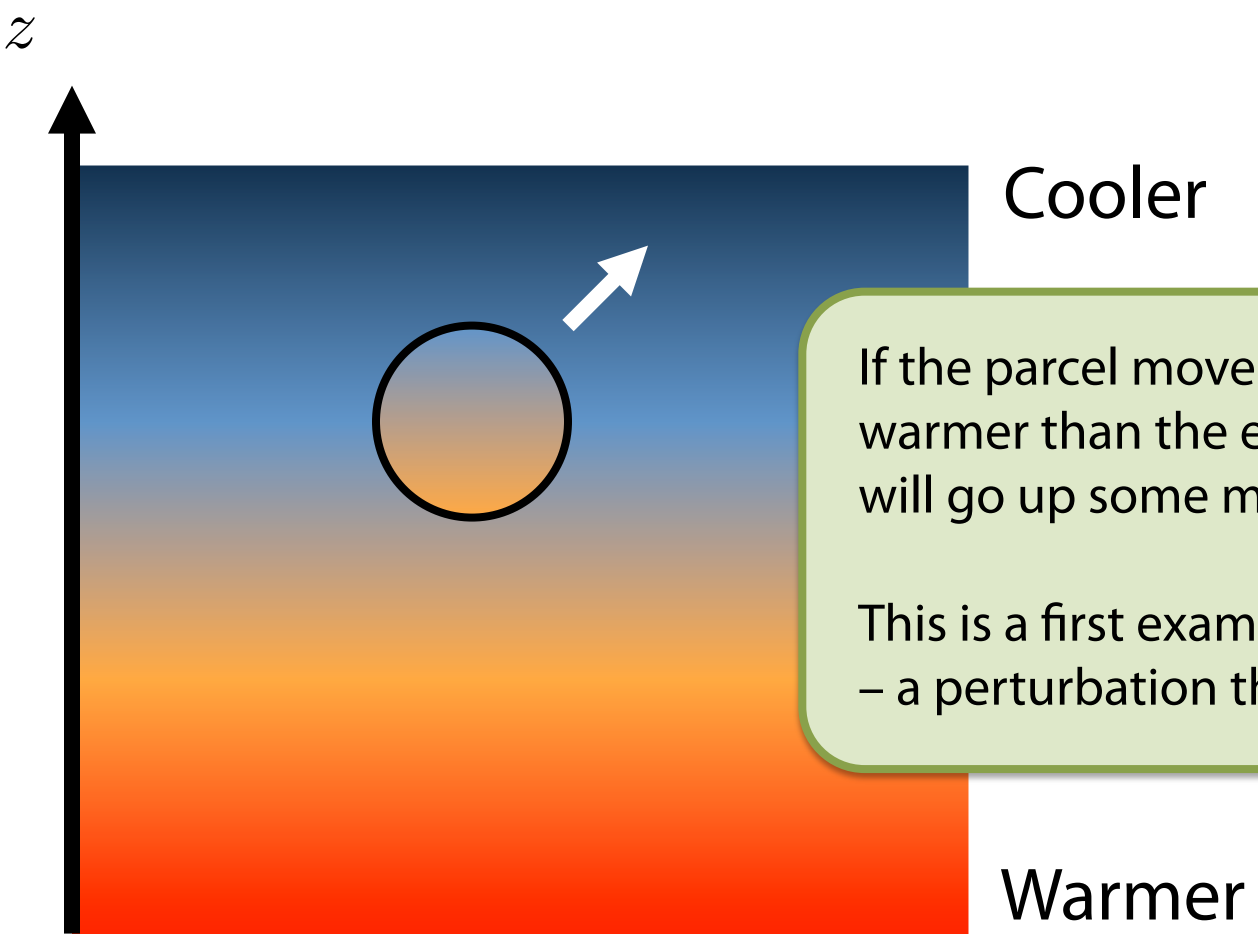
Aside: What is its density? Larger or smaller?

Dry Convection



Question: If the parcel moves up and finds itself warmer than the environment, what will happen?

Dry Convection



If the parcel moves up and finds itself warmer than the environment then it will go up some more.

This is a first example of "instability" – a perturbation that grows.

Dry Convection

Assume that temperature is a linear function of height:

$$T = T_s - \Gamma z \qquad \Gamma = -\frac{\partial T}{\partial z}$$

So if we go from z to $z + \Delta z$, then the change in T of the environment is

$$\Delta T = [T_s - \Gamma(z + \Delta z)] - [T_s - \Gamma z] = -\Gamma \Delta z$$

Dry Convection

If we go from z to $z + \Delta z$, then the change in T of the air parcel will follow the dry adiabatic lapse rate

$$\begin{aligned} \Delta T_p &= T_p(z + \Delta z) - T_p(z) \\ &= (T_p(z) - \Gamma_d \Delta z) - T_p(z) \\ &= -\Gamma_d \Delta z \end{aligned}$$

If we go from z to $z + \Delta z$, then the change in T of the environment will follow the environmental lapse rate

$$\begin{aligned} \Delta T_{env} &= T_{env}(z + \Delta z) - T_{env}(z) \\ &= (T_{env}(z) - \Gamma \Delta z) - T_{env}(z) \\ &= -\Gamma \Delta z \end{aligned}$$

Dry Convection

Stable: If the temperature of parcel is cooler than environment.

$$T_p < T_{env}$$

But since at the initial position of the parcel it has the same temperature as the environment,

$$\Delta T_p < \Delta T_{env}$$

Then using $\Delta T_p = -\Gamma_d \Delta z$ and $\Delta T_{env} = -\Gamma \Delta z$

$$-\Gamma_d \Delta z < -\Gamma \Delta z$$

In a stable atmospheric environment, the lapse rate satisfies $\Gamma < \Gamma_d$

Dry Convection

Unstable: If the temperature of parcel is warmer than environment.

$$T_p > T_{env}$$

But since at the initial position of the parcel it has the same temperature as the environment,

$$\Delta T_p > \Delta T_{env}$$

Then using $\Delta T_p = -\Gamma_d \Delta z$ and $\Delta T_{env} = -\Gamma \Delta z$

$$-\Gamma_d \Delta z > -\Gamma \Delta z$$

In an unstable atmospheric environment, the lapse rate satisfies $\Gamma > \Gamma_d$

Dry Convection

Stability criteria from physical argument

$$\Gamma_d > \Gamma \quad \textbf{Stable}$$

$$\Gamma_d = \Gamma \quad \textbf{Neutral}$$

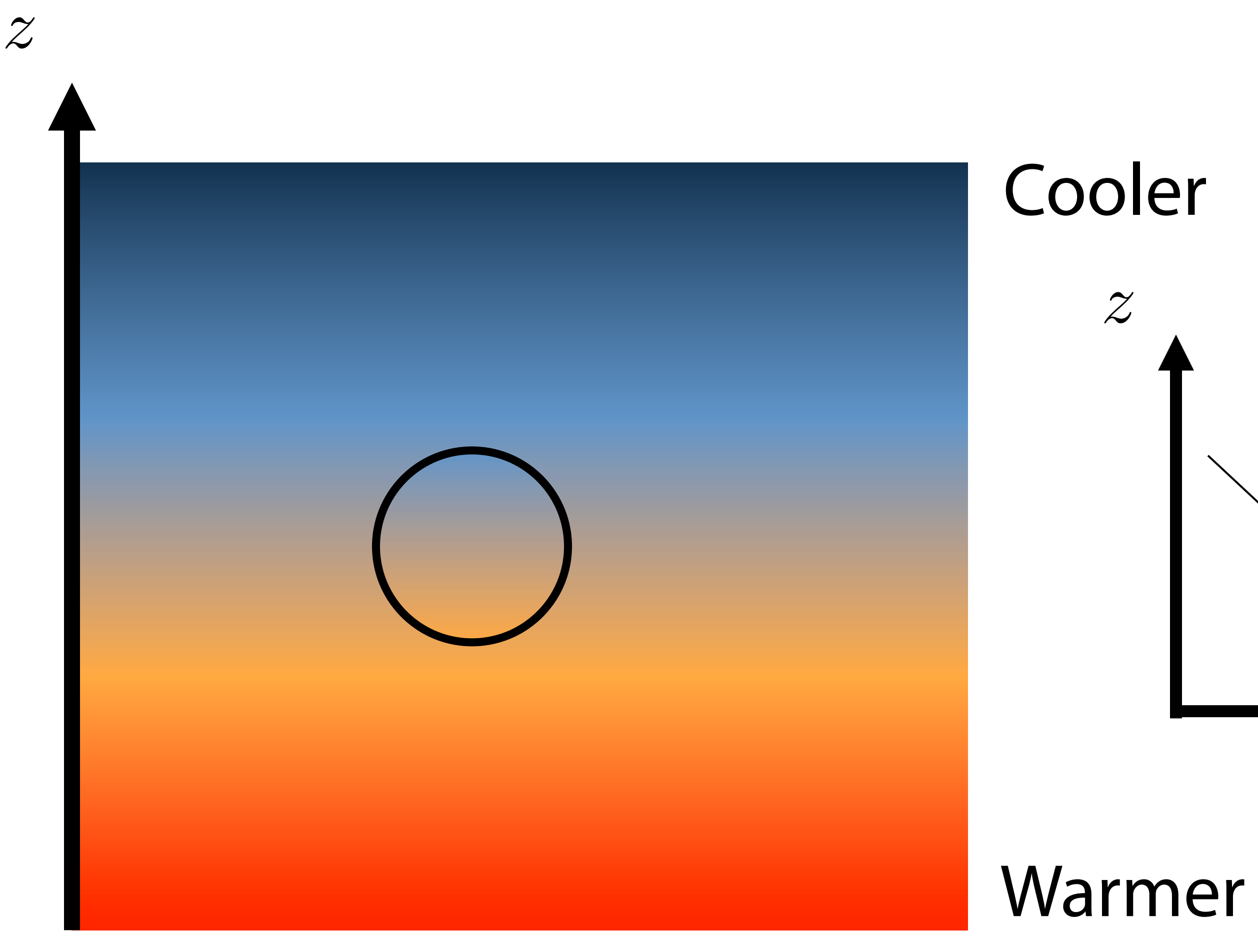
$$\Gamma_d < \Gamma \quad \textbf{Unstable}$$

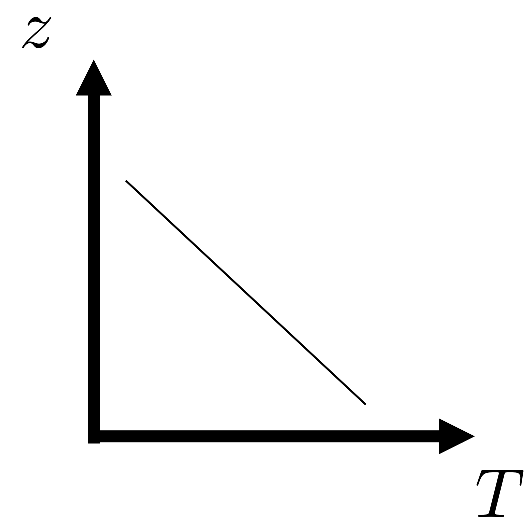
Adiabatic lapse rate

Environmental lapse rate

Important: A compressible atmosphere is unstable if temperature decreases with height faster than the adiabatic lapse rate.

Dry Convection





Dry Static Stability

Environment is in hydrostatic balance, no acceleration:

$$0 = -\frac{1}{\rho_{env}}\frac{\partial p_{env}}{\partial z} - g$$

But our parcel experiences an acceleration:

$$\frac{Dw}{Dt} = \frac{D^2 z}{Dt^2} = -\frac{1}{\rho_p}\frac{\partial p_p}{\partial z} - g$$

Assume instantaneous adjustment of parcel pressure

Dry Static Stability

But our parcel experiences an acceleration:

$$\frac{Dw}{Dt} = \frac{D^2 z}{Dt^2} = -\frac{g\rho_{env}}{\rho_p} - g = g\left(\frac{\rho_{env} - \rho_p}{\rho_p}\right)$$

$$\rho_{env} = \frac{p_{env}}{R_d T_{env}} \qquad \text{Ideal gas law} \qquad \rho_p = \frac{p_p}{R_d T_p}$$

$$\frac{D^2 z}{Dt^2} = g\left(\frac{T_p - T_{env}}{T_{env}}\right)$$

Dry Static Stability

$$\frac{D^2 z}{Dt^2} = g\left(\frac{T_p - T_{env}}{T_{env}}\right)$$

$$T_p = T(z_0) - \Gamma_d(z - z_0)$$

$$T_{env} = T(z_0) - \Gamma(z - z_0)$$

$$\frac{D^2 z}{Dt^2} = \frac{g}{T(z_0) - \Gamma(z - z_0)}(\Gamma - \Gamma_d)(z - z_0)$$

Binomial expansion

$$\frac{1}{T(z_0) - \Gamma(z - z_0)} = \frac{1}{T(z_0)} \frac{1}{1 - \frac{\Gamma}{T(z_0)}(z - z_0)}$$

$$\approx \frac{1}{T(z_0)}\left(1 + \frac{\Gamma(z - z_0)}{T(z_0)}\right)$$

Dry Static Stability

$$\frac{D^2 z}{Dt^2} = \frac{g}{T(z_0)}(\Gamma - \Gamma_d)(z - z_0) + \frac{g\Gamma}{T(z_0)^2}(z - z_0)^2$$

For small displacements ignore quadratic terms:

$$\frac{D^2 z}{Dt^2} \approx \frac{g}{T(z_0)}(\Gamma - \Gamma_d)(z - z_0)$$

Define $\Delta z = z - z_0$

$$\frac{D^2 \Delta z}{Dt^2} + \frac{g}{T(z_0)}(\Gamma_d - \Gamma)\Delta z = 0$$

Dry Static Stability

$$\frac{D^2 \Delta z}{Dt^2} + \frac{g}{T(z_0)}(\Gamma_d - \Gamma)\Delta z = 0$$

This is an ordinary differential equation in the variable Δz.
Do you recognize this equation and its solutions?

Three cases:

$$\left[\begin{array}{l} \dfrac{g(\Gamma_d - \Gamma)}{T(z_0)} > 0 \\[2ex] \dfrac{g(\Gamma_d - \Gamma)}{T(z_0)} = 0 \\[2ex] \dfrac{g(\Gamma_d - \Gamma)}{T(z_0)} < 0 \end{array}\right.$$

These cases correspond to stable, neutral and unstable atmospheres.

Stable Conditions

$$\frac{D^2 \Delta z}{Dt^2} + \frac{g}{T(z_0)}(\Gamma_d - \Gamma)\Delta z = 0$$

$$\frac{g(\Gamma_d - \Gamma)}{T(z_0)} > 0$$

Stable atmosphere

Definition: The **Brunt-Väisälä Frequency** is the frequency of an oscillating air parcel in a stable atmosphere:

$$\mathcal{N}^2 = \frac{g(\Gamma_d - \Gamma)}{T(z_0)}$$

Units of seconds^2

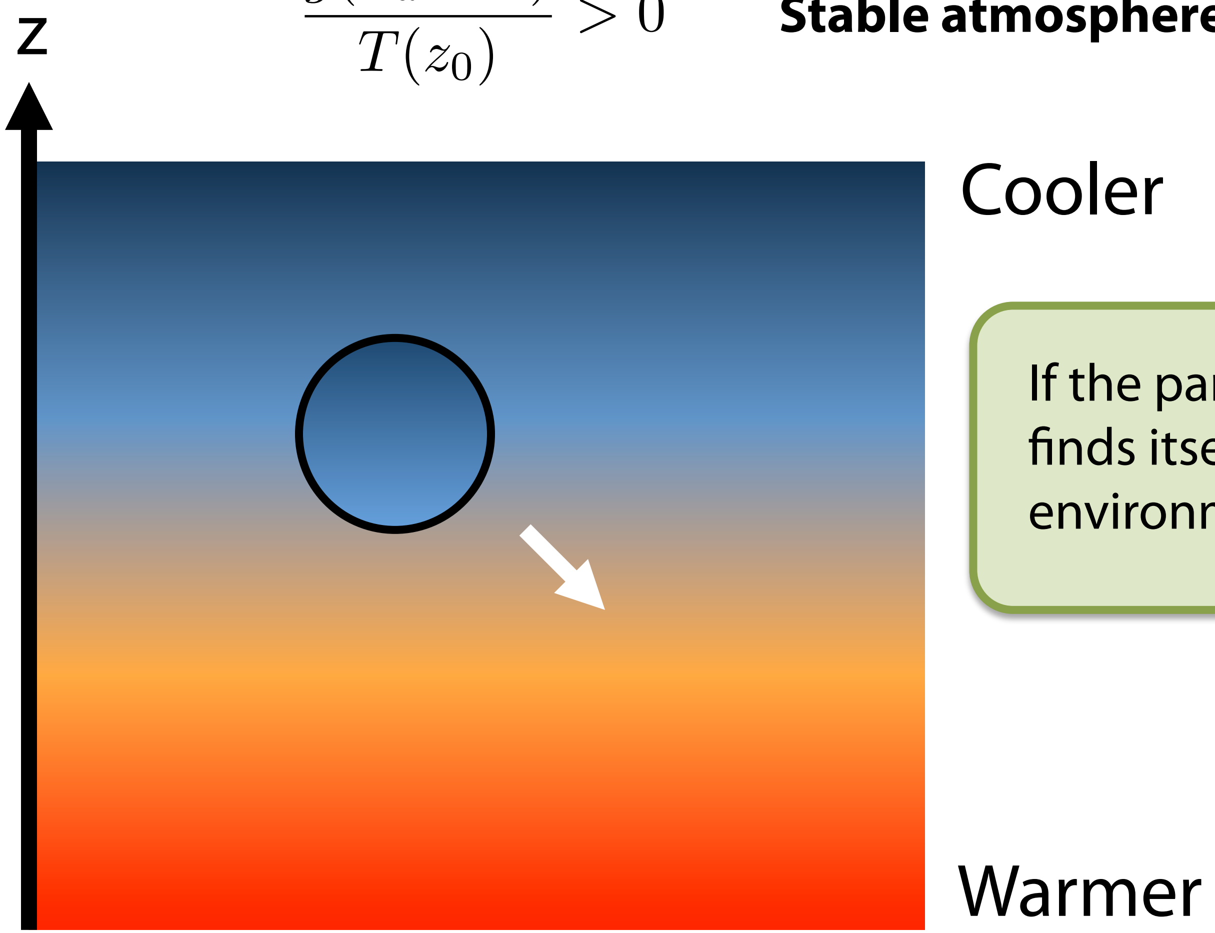
$$\Delta z = \Delta z_0 \sin\left(\mathcal{N} t + \phi\right)$$

Oscillatory solutions (magnitude of oscillation depends on initial velocity)

Stable Conditions

$$\frac{g(\Gamma_d - \Gamma)}{T(z_0)} > 0$$

Stable atmosphere



If the parcel moves up and finds itself cooler than the environment then it will sink.

Stable Conditions

Brunt-Väisälä Frequency

$$\mathcal{N}^2 = \frac{g(\Gamma_d - \Gamma)}{T(z_0)}$$

Brunt-Väisälä Frequency

$$\mathcal{N}^2 = \frac{g}{\theta}\frac{\partial \theta}{\partial z}$$

Exercise: These two forms are equivalent.

Stable Conditions

Example of such an oscillation:

Stable Conditions

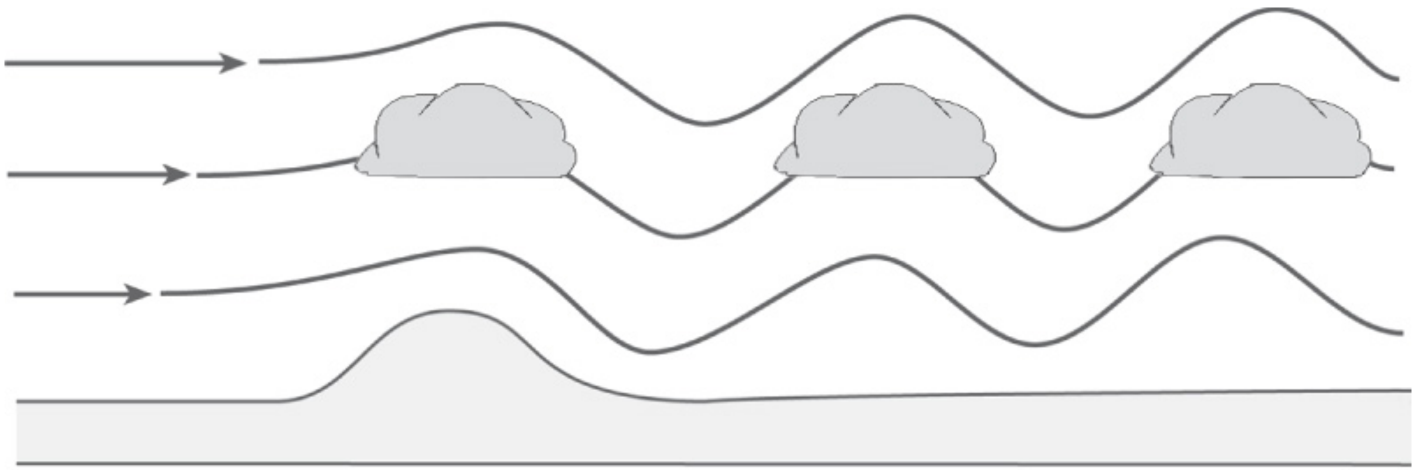


Figure: A schematic diagram illustrating the formation of mountain waves (also known as lee waves). The presence of the mountain disturbs the air flow and produces a train of downstream waves.

Directly over the mountain, a distinct cloud type known as lenticular ("lens-like") cloud is frequently produced.

Downstream and aloft, cloud bands may mark parts of the wave train in which air has been uplifted (and thus cooled to saturation).

Stable Conditions

Figure: A lenticular ("lens-like") cloud.

Neutral Conditions

$$\frac{D^2 \Delta z}{Dt^2} + \frac{g}{T(z_0)}(\Gamma_d - \Gamma)\Delta z = 0$$

$$\frac{g(\Gamma_d - \Gamma)}{T(z_0)} = 0$$

Neutral atmosphere

$$\Delta z = u_0 t$$

Parcel does not experience acceleration; travels at initial velocity.

Unstable Conditions

$$\frac{D^2\Delta z}{Dt^2} + \frac{g}{T(z_0)}(\Gamma_d - \Gamma)\Delta z = 0$$

$$\frac{g(\Gamma_d - \Gamma)}{T(z_0)} < 0$$ **Unstable atmosphere**

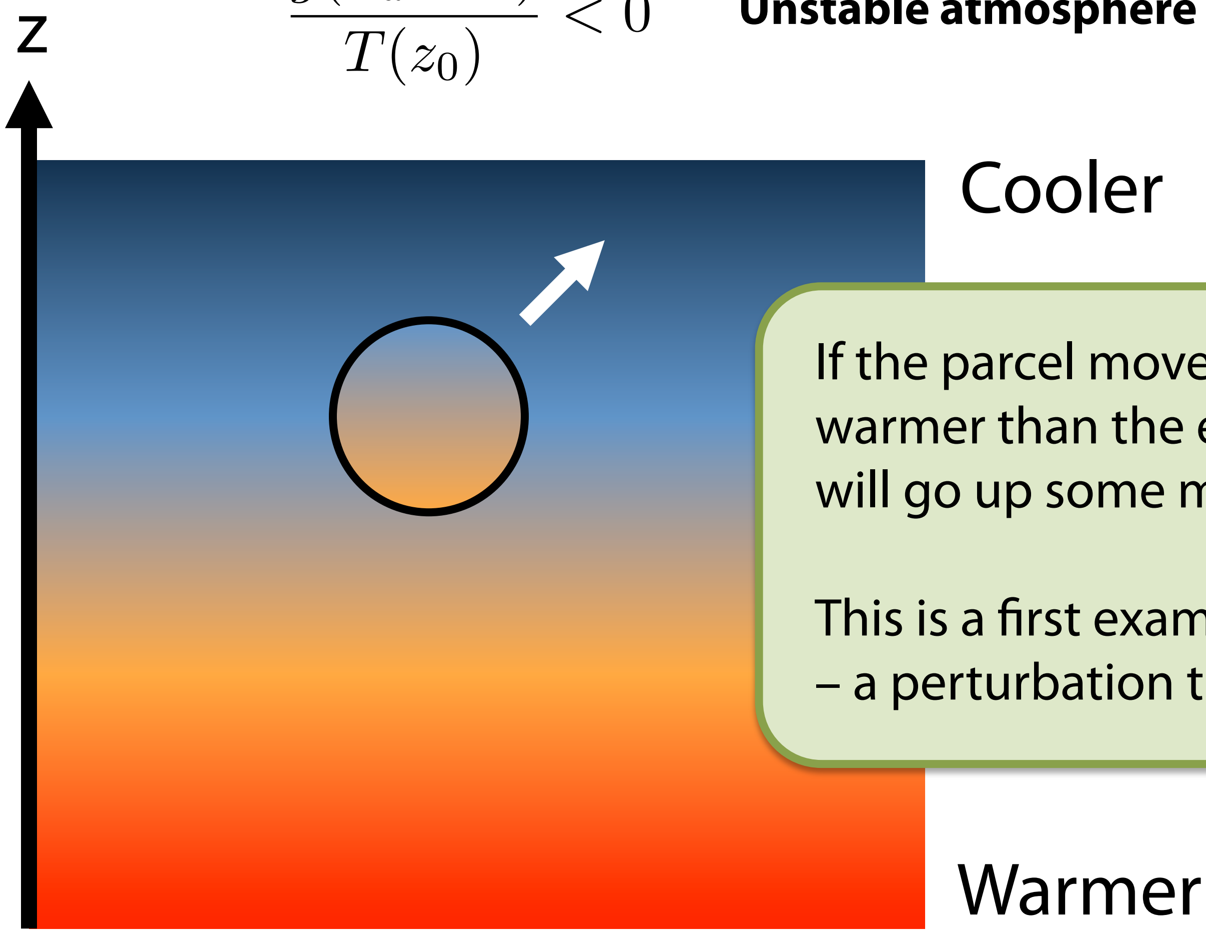
$$\Delta z = A\exp\left(\frac{g(\Gamma - \Gamma_d)}{T(z_0)}t\right) + B\exp\left(-\frac{g(\Gamma - \Gamma_d)}{T(z_0)}t\right)$$

Exponential solutions (at least one of these terms will grow without bound).

Unstable Conditions

$$\frac{g(\Gamma_d - \Gamma)}{T(z_0)} < 0$$

Unstable atmosphere



If the parcel moves up and finds itself warmer than the environment then it will go up some more.

This is a first example of "instability" – a perturbation that grows.

Unstable Conditions

Static Stability

Brunt-Väisälä Frequency

$$\mathcal{N}^2 = g\frac{\partial(\ln\theta)}{\partial z} = \frac{g}{\theta}\frac{\partial\theta}{\partial z}$$

Exercise: Show this definition of Brunt-Väisälä frequency is equivalent to the previous definition.

$$\begin{cases} \dfrac{\partial\theta}{\partial z} > 0 & \textbf{Statically Stable} \\ \dfrac{\partial\theta}{\partial z} = 0 & \textbf{Statically Neutral} \\ \dfrac{\partial\theta}{\partial z} < 0 & \textbf{Statically Unstable} \end{cases}$$

Static Stability

Two ways of determining stability:

$$\begin{cases} \Gamma < \Gamma_d \Leftrightarrow \dfrac{\partial \theta}{\partial z} > 0 & \textbf{Statically Stable} \\ \Gamma = \Gamma_d \Leftrightarrow \dfrac{\partial \theta}{\partial z} = 0 & \textbf{Statically Neutral} \\ \Gamma > \Gamma_d \Leftrightarrow \dfrac{\partial \theta}{\partial z} < 0 & \textbf{Statically Unstable} \end{cases}$$