Analysis of the Dynamical Equations

Chapter 2

Paul A. Ullrich

paullrich@ucdavis.edu

Part 2: Scale Analysis and Ageostropic Wind

Scale Analysis

Question: What are the terms in the continuity equation that are most relevant for large-scale mid-latitude dynamics?

As we will see, this question is closely connected to a question that arose last time:

Question: How do we compute vertical velocity without using the vertical momentum equation?

Starting point: Consider a "background" state for the thermodynamic variables.

$$p_0 = p_0(z) \qquad \rho_0 = \rho_0(z)$$

No variation in x, y, t

This represents an "average" of these variables at each level of the atmosphere.

Hydrostatic balance applies to this state:

$$\frac{dp_0}{dz} = -\rho_0 g$$

Next, define a "perturbation" from the background:

$$p'(t,x,y,z) = p(t,x,y,z) - p_0(z)$$
$$\rho'(t,x,y,z) = \rho(t,x,y,z) - \rho_0(z)$$

Continuity Equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

Eulerian Frame

Chain Rule $\Rightarrow$ $$\frac{\partial \rho}{\partial t} = -\rho \nabla \cdot \mathbf{u} - \mathbf{u} \cdot \nabla \rho$$

$\rho = \rho_0 + \rho'$ $\Rightarrow$ $$\cancelto{0}{\frac{\partial \rho_0}{\partial t}} + \frac{\partial \rho'}{\partial t} = -\rho_0 \nabla \cdot \mathbf{u} - \rho' \nabla \cdot \mathbf{u} - \mathbf{u} \cdot \nabla \rho_0 - \mathbf{u} \cdot \nabla \rho'$$

Observe $$\mathbf{u} \cdot \nabla \rho_0 = u \cancelto{0}{\frac{\partial \rho_0}{\partial x}} + v \cancelto{0}{\frac{\partial \rho_0}{\partial y}} + w \frac{\partial \rho_0}{\partial z}$$

$\Rightarrow$ $$\left[\frac{\partial \rho'}{\partial t} + \mathbf{u} \cdot \nabla \rho' \right] = -\rho_0 \nabla \cdot \mathbf{u} - \rho' \nabla \cdot \mathbf{u} - w \frac{\partial \rho_0}{\partial z}$$

$$\boxed{\left[\frac{\partial \rho'}{\partial t} + \mathbf{u}\cdot\nabla\rho'\right] = -\rho_0 \nabla\cdot\mathbf{u} - \rho'\nabla\cdot\mathbf{u} - w\frac{\partial \rho_0}{\partial z}}$$

Divide by ρ_0 and assume $\rho'/\rho_0 \approx 10^{-2} \ll 1$

$$\frac{1}{\rho_0}\left[\frac{\partial \rho'}{\partial t} + \mathbf{u}\cdot\nabla\rho'\right] = -\cancelto{\text{small}}{\frac{\rho'}{\rho_0}\nabla\cdot\mathbf{u}} - \cancelto{1}{\frac{\rho_0}{\rho_0}}\nabla\cdot\mathbf{u} - \frac{w}{\rho_0}\frac{d\rho_0}{dz}$$

$$\frac{1}{\rho_0}\left[\frac{\partial \rho'}{\partial t} + \mathbf{u}\cdot\nabla\rho'\right] + \nabla\cdot\mathbf{u} + \frac{w}{\rho_0}\frac{d\rho_0}{dz} = 0$$

Now analyze scales…

(Continuity Equation)

Scales

$$U \approx 10 \text{ m s}^{-1}$$
$$W \approx 0.01 \text{ m s}^{-1}$$
$$L \approx 10^6 \text{ m}$$
$$H \approx 10^4 \text{ m}$$
$$L/U \approx 10^5 \text{ s}$$

$$\Delta P \approx 1000 \text{ Pa}$$
$$\rho \approx 1 \text{ kg m}^{-3}$$
$$\Delta\rho/\rho \approx 10^{-2}$$
$$f_0 \approx 10^{-4} \text{ s}^{-1}$$

$$a \approx 10^7 \text{ m}$$
$$g \approx 10 \text{ m s}^{-2}$$
$$\nu \approx 10^{-5} \text{ m}^2 \text{ s}^{-1}$$

$$\frac{1}{\rho_0}\left[\frac{\partial \rho'}{\partial t} + \mathbf{u}\cdot\nabla\rho'\right] + \nabla\cdot\mathbf{u} + \frac{w}{\rho_0}\frac{d\rho_0}{dz} = 0$$

$\Delta\rho/\rho \cdot U/L$	U/L	W/H
10^{-7} s^{-1}	10^{-5} s^{-1} ?	10^{-6} s^{-1}

Nothing to balance divergence term?

Look more closely at the divergence term…

$$\frac{1}{\rho_0}\left[\frac{\partial \rho'}{\partial t} + \mathbf{u}\cdot\nabla\rho'\right] + \nabla\cdot\mathbf{u} + \frac{w}{\rho_0}\frac{d\rho_0}{dz} = 0$$

$$\frac{1}{\rho_0}\left[\frac{\partial \rho'}{\partial t} + \mathbf{u}\cdot\nabla\rho'\right] + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \frac{w}{\rho_0}\frac{d\rho_0}{dz} = 0$$

$$\frac{\Delta\rho}{\rho}\cdot\frac{U}{L} \sim 10^{-7}s^{-1}$$

These two terms scale as U/L, but with opposite sign

$$\sim \cancel{10^{-5}s^{-1}}\ 10^{-6}s^{-1}$$

$$\frac{W}{H} \sim 10^{-6}s^{-1}$$

Balance is now between horizontal divergence and vertical transport of ρ_0

small

$$\frac{1}{\rho_0}\left[\cancel{\frac{\partial \rho'}{\partial t} + \mathbf{u}\cdot\nabla\rho'}\right] + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \frac{w}{\rho_0}\frac{d\rho_0}{dz} = 0$$

Dominant balance is between these terms

Since horizontal variations of ρ_0 are zero and we can add them back in

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \frac{1}{\rho_0}\left(u\frac{\partial \rho_0}{\partial x} + v\frac{\partial \rho_0}{\partial y} + w\frac{\partial \rho_0}{\partial w}\right) = 0$$

$$\boxed{\nabla\cdot(\rho_0\mathbf{u}) = 0}$$

Where does this lead us?

$$\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+\frac{w}{\rho_0}\frac{d\rho_0}{dz}\approx 0$$

$$\Rightarrow \quad \rho_0\frac{\partial w}{\partial z}+w\frac{\partial \rho_0}{\partial z}\approx -\rho_0\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

But by product rule $\dfrac{\partial(\rho_0 w)}{\partial z}=\rho_0\dfrac{\partial w}{\partial z}+w\dfrac{\partial \rho_0}{\partial z}$

$$\Rightarrow \quad \frac{\partial(\rho_0 w)}{\partial z}\approx -\rho_0\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

$$\Rightarrow \quad \boxed{w\approx -\frac{1}{\rho_0}\int_0^z \rho_0\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)dz'}$$

Large-scale (synoptic) vertical motions are proportional to the vertically integrated **horizontal divergence.**

$$w \approx -\frac{1}{\rho_0}\int_0^z \rho_0 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dz'$$

Diagnostic equation for **w** from the **horizontal divergence**.

Large-scale (synoptic) vertical motions are proportional to the vertically integrated **horizontal divergence.**

We can look for where there is divergence of horizontal wind in the middle to upper troposphere and diagnose vertical motion in the underlying atmosphere…

So far, we have used scale analysis...

- To show that the dominant terms in the horizontal momentum equations correspond to **geostrophic balance** (horizontal pressure gradient force and Coriolis force).
- To show that the dominant terms in the vertical momentum equations correspond to **hydrostatic balance** (vertical pressure gradient force and gravity).
- To show that vertical velocity can be **diagnosed** from vertically integrated horizontal divergence.

Question: All of these results are **diagnostic**. How do we actually **predict** the evolution of the atmosphere?

(Horizontal Momentum Equation)

Largest Terms

$$\frac{Du}{Dt} - \frac{uv\tan\phi}{r} + \frac{uw}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2\Omega v\sin\phi - 2\Omega w\cos\phi + \nu\nabla^2 u$$

$$\frac{Dv}{Dt} + \frac{u^2\tan\phi}{r} + \frac{vw}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2\Omega u\sin\phi + \nu\nabla^2 v$$

U·U/L	U·U/a	U·W/a	ΔP/ρL	Uf	Wf	$\nu U/H^2$
10^{-4}	10^{-5}	10^{-8}	10^{-3}	10^{-3}	10^{-6}	10^{-12}

Dominant Balance

Geostrophic Balance

Definition: To simplify notation, the **Coriolis parameter** is defined as $f = 2\Omega \sin\phi$

Geostrophic Balance

$$\begin{aligned} \frac{1}{\rho}\frac{\partial p}{\partial x} &= \quad 2\Omega v \sin\phi = \quad f v \\ \frac{1}{\rho}\frac{\partial p}{\partial y} &= -2\Omega u \sin\phi = -f u \end{aligned}$$

There is no **D/Dt** term – no acceleration, no change with time.

This is a **DIAGNOSTIC** equation that can be used to analyze how the atmosphere works.

Geostrophic Wind

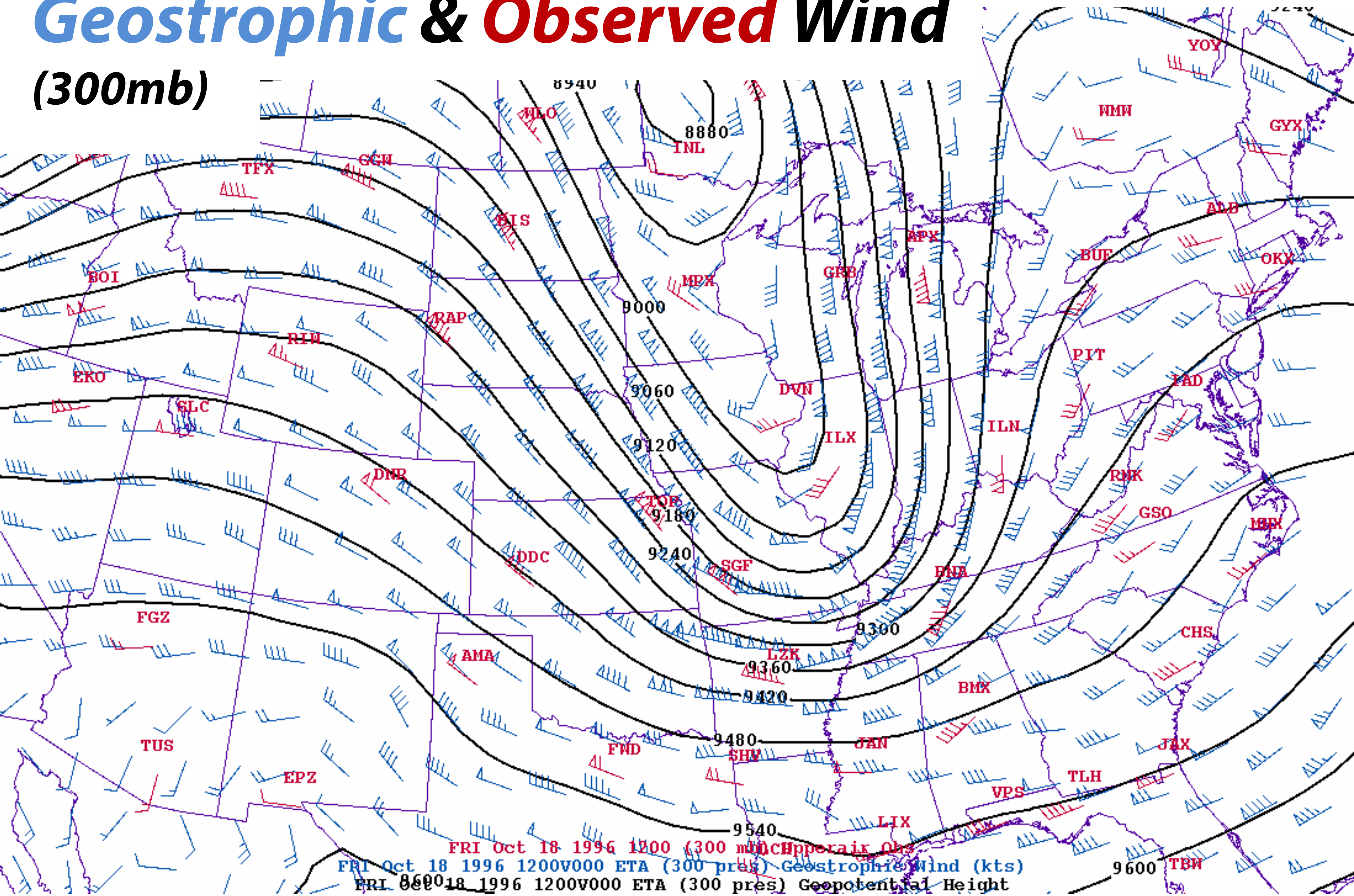
The notion of geostrophic balance motivates a natural decomposition of the horizontal velocity vector u into geostrophically balanced and non-geostrophically balanced parts.

Definition: The **geostrophic wind** is the component of the real wind which is governed by geostrophic balance. On constant height surfaces, it is defined to satisfy

$$u_g = -\frac{1}{f\rho}\frac{\partial p}{\partial y}$$

$$v_g = +\frac{1}{f\rho}\frac{\partial p}{\partial x}$$

Geostrophic & Observed Wind
(300mb)



Recall our diagnostic equation for w…

$$w \approx -\frac{1}{\rho_0} \int_0^z \rho_0 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dz'$$

Question: If the flow is geostrophic, what is the vertical velocity?

$$f = 2\Omega \sin\phi$$

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = 2\Omega v \sin\phi = fv$$

$$\frac{1}{\rho}\frac{\partial p}{\partial y} = -2\Omega u \sin\phi = -fu$$

Assume that you look at a spatial scale that is small enough that f is constant and density is a function of height only.

What is the divergence of the velocity in this case?

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = 2\Omega v \sin\phi = fv$$

$$\frac{1}{\rho}\frac{\partial p}{\partial y} = -2\Omega u \sin\phi = -fu$$

Approximations…

$$\frac{\partial \rho}{\partial x} = \frac{\partial \rho}{\partial y} = 0$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$$

$$\frac{1}{f\rho}\frac{\partial}{\partial y}\frac{\partial p}{\partial x} = \frac{\partial v}{\partial y}$$

$$-\frac{1}{f\rho}\frac{\partial}{\partial x}\frac{\partial p}{\partial y} = \frac{\partial u}{\partial x}$$

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{1}{f\rho}\left[\frac{\partial^2 p}{\partial x \partial y} - \frac{\partial^2 p}{\partial y \partial x}\right] = 0}$$

For **geostrophic motion** (and negligible variation of ρ and f on a horizontal surface), the **horizontal divergence is effectively zero**.

For **geostrophic motion** (and negligible variation of ρ and f on a horizontal surface), the **horizontal divergence is effectively zero**.

Recall our diagnostic equation for vertical velocity:

$$w \approx -\frac{1}{\rho_0} \int_0^z \rho_0 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dz'$$

If horizontal divergence is effectively zero, then so is vertical velocity.

For **geostrophic motion** (and negligible variation of ρ and f on a horizontal surface), the **vertical velocity is effectively zero**.

We are faced with a difficult situation…

The middle latitude atmosphere is in a state of near balance (geostrophic and hydrostatic). BUT…

- There is no vertical motion associated with geostrophic flow
- There is no acceleration associated with geostrophic flow

Recall the question we are trying to answer:

Question: All of these results are **diagnostic**. How do we actually **predict** the evolution of the atmosphere?

Result: The largest terms in the equations of motion correspond to an atmosphere in balance.

Large-scale weather systems (the phenomena that needs to be forecasted) correspond to **small differences** from the balanced state.

(Horizontal Momentum Equation)

$$\frac{Du}{Dt} - \frac{uv\tan\phi}{r} + \frac{uw}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2\Omega v\sin\phi - 2\Omega w\cos\phi + \nu\nabla^2 u$$

$$\frac{Dv}{Dt} + \frac{u^2\tan\phi}{r} + \frac{vw}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2\Omega u\sin\phi + \nu\nabla^2 v$$

U·U/L	U·U/a	U·W/ a	ΔP/ρL	Uf	Wf	$\nu U/H^2$
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Prediction (Prognosis)
Ageostrophic

Analysis (Diagnostic)
Geostrophic

Our prediction equation for large-scale mid-latitudinal forecasting

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv$$

$$\frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu$$

Difference of real wind from geostrophic wind

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv = \quad f\left(v - \frac{1}{f\rho}\frac{\partial p}{\partial x}\right) = \quad f(v - v_g)$$

$$\frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu = -f\left(u + \frac{1}{f\rho}\frac{\partial p}{\partial y}\right) = -f(u - u_g)$$

We used the definition of the **geostrophic component** of the wind, which is within 10-15% of the real wind in middle latitudes (for large-scale motion)

$$v_g = \quad \frac{1}{f\rho}\frac{\partial p}{\partial x}$$

$$u_g = -\frac{1}{f\rho}\frac{\partial p}{\partial y}$$

For middle latitudes and large scales, the acceleration can be computed directly as the **difference from geostrophic balance**.

Our prediction equation for large-scale mid-latitudinal forecasting

$$\frac{Du}{Dt} = f(v - v_g)$$
$$\frac{Dv}{Dt} = -f(u - u_g)$$

$$v_g = \frac{1}{f\rho}\frac{\partial p}{\partial x}$$
$$u_g = -\frac{1}{f\rho}\frac{\partial p}{\partial y}$$

Definition: The **ageostrophic wind** is defined as the difference between the real wind and geostrophic wind.

$$\mathbf{u}_{ag} = \mathbf{u} - \mathbf{u}_g$$

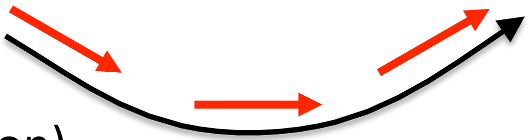
Remember: pressure and density are buried inside the definition of the geostrophic wind. The **mass field** and **velocity field** are linked.

Ageostrophic Wind

$$\frac{Du}{Dt} = f(v - v_g)$$
$$\frac{Dv}{Dt} = -f(u - u_g)$$

Acceleration is proportional to the difference between the real wind and the geostrophic wind.

Acceleration can be categorized into **two types**:

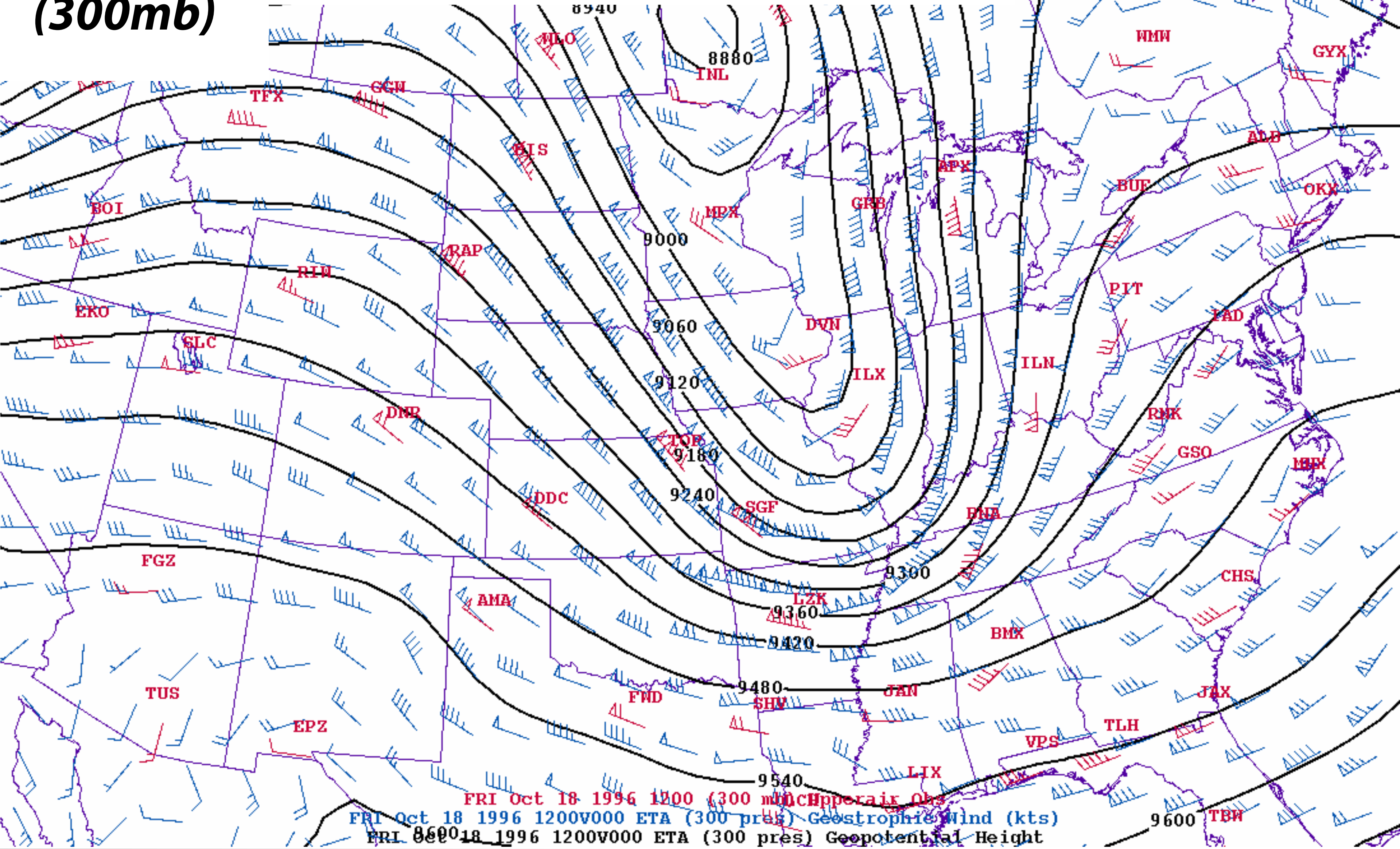


(a) Change in direction of the flow (curvature/rotation)

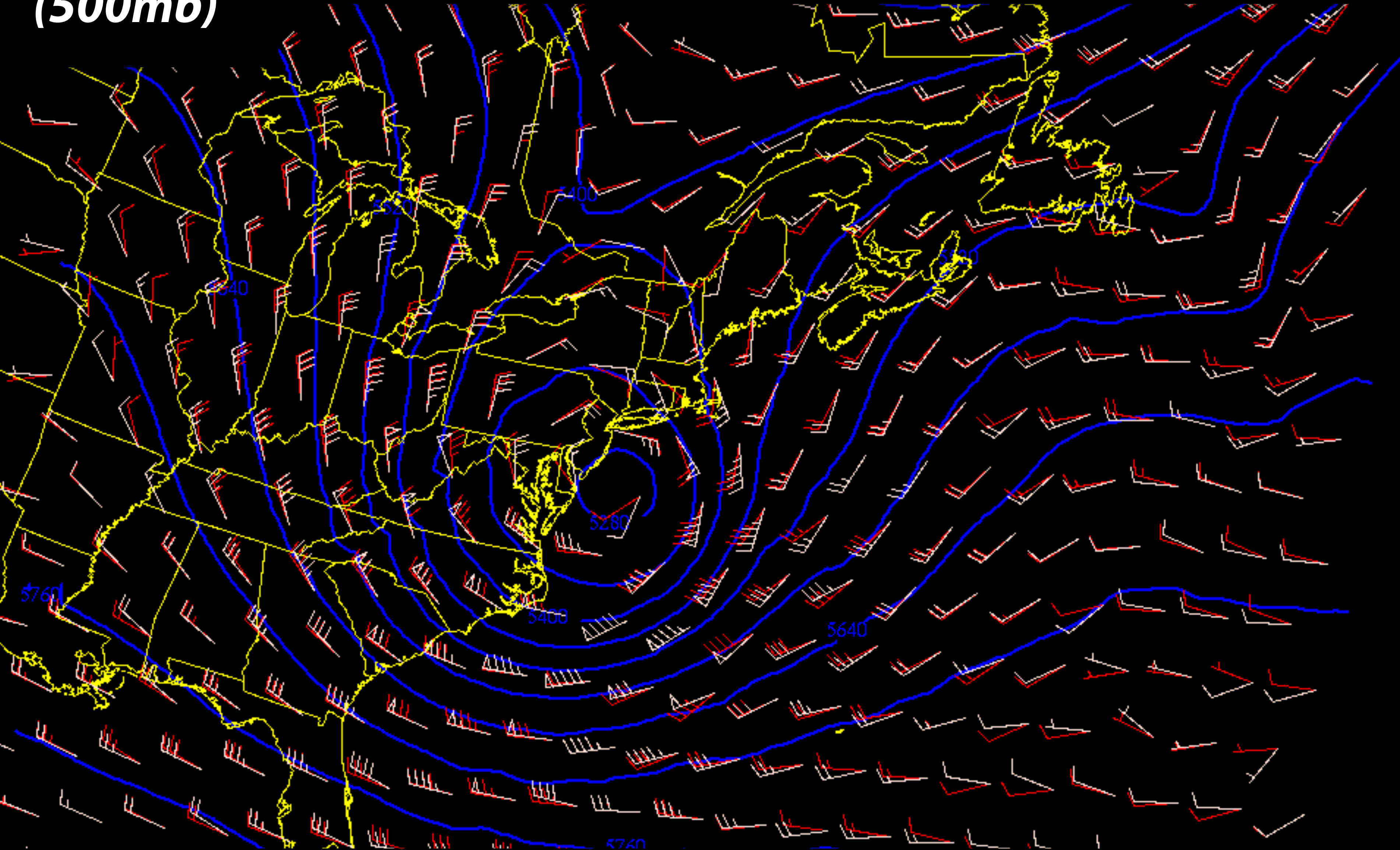
(b) Along-flow change in speed (convergence/divergence)

Geostrophic & Observed Wind

(300mb)



Geostrophic & Observed Wind
(500mb)



Ageostrophic Wind

Remember the scaled continuity equation…

These scale as U/L but with opposite sign.

$\sim 10^{-6}\ \mathrm{s}^{-1}$

$$\left(\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}} + \frac{\partial w}{\partial z}\right) + \frac{w}{\rho_0}\frac{d\rho_0}{dz} = 0$$

Vertical motion is related to divergence, but **geostrophic wind** is **essentially nondivergent**.

Hence, the divergence of the ageostrophic wind must be the driver of vertical motion on large scales

Question: What does this picture tell us about vertical velocity?

Question: What does this picture tell us about our scale analysis?

Geostrophic Wind

Observed Wind

Boundary Layer Observations
Effect on winds near the surface