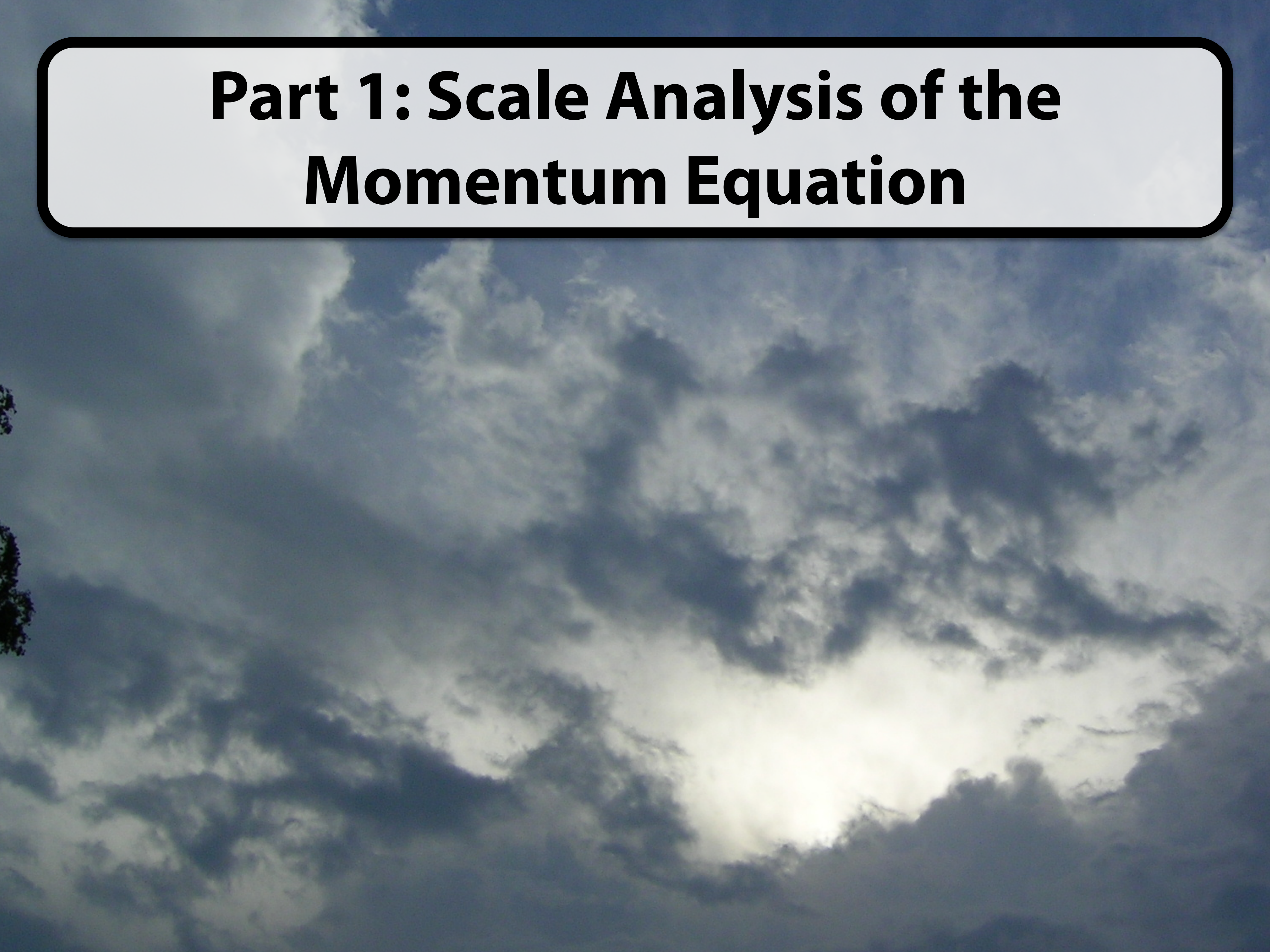
Analysis of the Dynamical Equations

Chapter 2

Paul A. Ullrich

paullrich@ucdavis.edu

Part 1: Scale Analysis of the Momentum Equation



The Atmospheric Equations

$$\frac{Du}{Dt} - \frac{uv\tan\phi}{r} + \frac{uw}{r} = -\frac{1}{\rho r\cos\phi}\frac{\partial p}{\partial \lambda} + 2\Omega v\sin\phi - 2\Omega w\cos\phi + \nu\nabla^2 u$$

$$\frac{Dv}{Dt} + \frac{u^2\tan\phi}{r} + \frac{vw}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \phi} - 2\Omega u\sin\phi + \nu\nabla^2 v$$

$$\frac{Dw}{Dt} - \frac{u^2+v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} - g + 2\Omega u\cos\phi + \nu\nabla^2 w$$

$$\frac{D\rho}{Dt} = -\rho\cdot\nabla\mathbf{u}$$

$$c_p\frac{DT}{Dt} - \frac{1}{\rho}\frac{Dp}{Dt} = J$$

$$p = \rho R_d T$$

$$\frac{Dq_i}{Dt} = S_i$$

Material derivative:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla$$

Scale Analysis

Question: What is the size (in time and space) of atmospheric phenomena that are relevant for large-scale mid-latitude dynamics?

Question: What are the terms in the equations of motion that are most relevant for large-scale mid-latitude dynamics?

These questions are closely connected: **Scale analysis** provides us a means to an answer.

Scale Analysis

Consider x and y components of the momentum equations:

$$\frac{Du}{Dt} - \frac{uv\tan\phi}{r} + \frac{uw}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2\Omega v\sin\phi - 2\Omega w\cos\phi + \nu\nabla^2 u$$

$$\frac{Dv}{Dt} + \frac{u^2\tan\phi}{r} + \frac{vw}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2\Omega u\sin\phi + \nu\nabla^2 v$$

Remember the units—each term must have units of ***acceleration***

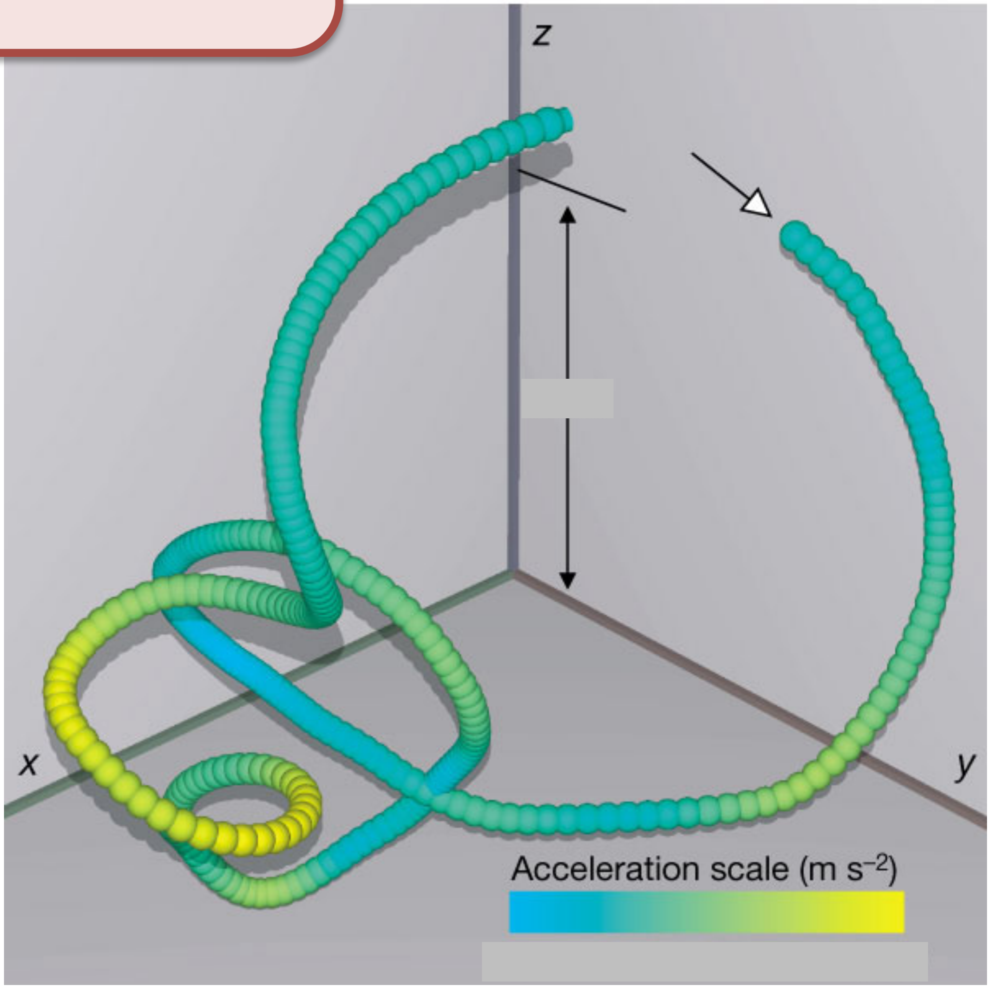
Define:	L	Some characteristic distance
	T	Some characteristic time

All terms should have units of L/T^2

Scale Analysis

Question: How do we define a time scale?

Idea: Take a ***typical*** trajectory and ask "how far does a parcel go in a 'characteristic time?'"



Scale Analysis

We would like to define scales in terms of wind, pressure and density.

Recall: $$\boxed{\langle\text{Distance}\rangle = \langle\text{Velocity}\rangle \times \langle\text{Time}\rangle}$$

Define: U Some characteristic horizontal velocity

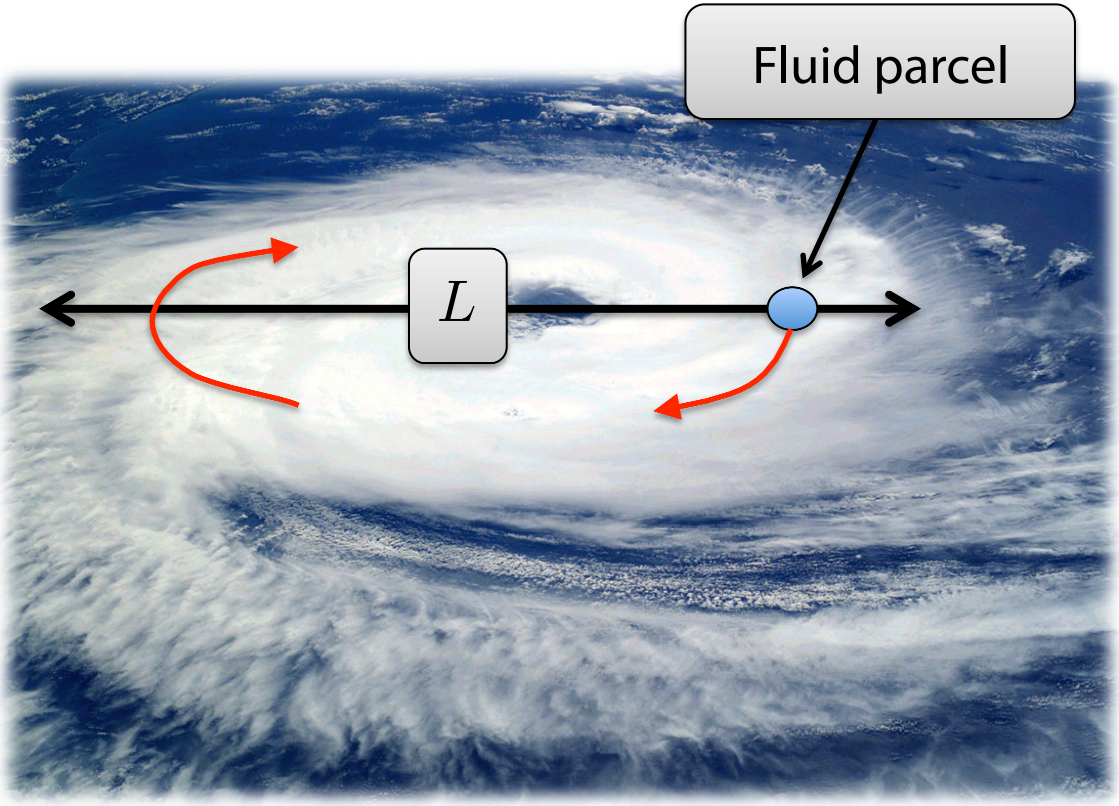
In terms of characteristic scales: $L = U \times T$

So the characteristic time scale is given by $$\boxed{T = \frac{L}{U}}$$

Scale Analysis

What do these scales mean?

Consider, for example, a typical tropical cyclone:



Look at the organization of the flow. The **length scale** is the diameter of the storm. The velocity scale is the **maximum velocity** of the flow. So the time scale is approximately the **time required for the parcel to move around the storm**.

Scale Analysis

Definition:	L	Horizontal distance scale
	U	Horizontal velocity scale
	T	Time scale $(T = L/U)$
	H	Vertical distance scale
	W	Vertical velocity scale
	$\Delta P/\rho$	Scale of horizontal pressure fluctuations

The material derivative $\frac{D}{Dt}$ represents change on the time-scale of motion.

Hence, its scale is given by $$\left[\frac{D}{Dt}\right] = \frac{1}{T} = \frac{U}{L}$$

Scale Analysis

Typical scales associated with large-scale mid-latitude storm systems:

$$U \approx 10 \text{ m s}^{-1}$$

$$W \approx 0.01 \text{ m s}^{-1}$$

$$L \approx 10^6 \text{ m}$$

$$H \approx 10^4 \text{ m}$$

$$L/U \approx 10^5 \text{ s}$$

$$\Delta P \approx 10 \text{ hPa} = 1000 \text{ Pa}$$

$$\rho \approx 1 \text{ kg m}^{-3}$$

$$\Delta\rho/\rho \approx 10^{-2}$$

$$f_0 \approx 10^{-4} \text{ s}^{-1}$$

$a \approx 10^7 \text{ m}$ (Radius of Earth)

$g \approx 10 \text{ m s}^{-2}$ (Gravity)

$\nu \approx 10^{-5} \text{ m}^2 \text{ s}^{-1}$ (Kinematic Viscosity)

Scale Analysis *(Horizontal Momentum)*

Here is what each term of the momentum equation looks like in terms of characteristic scales:

$$\frac{Du}{Dt} - \frac{uv\tan\phi}{r} + \frac{uw}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2\Omega v\sin\phi - 2\Omega w\cos\phi + \nu\nabla^2 u$$

$$\frac{Dv}{Dt} + \frac{u^2\tan\phi}{r} + \frac{vw}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2\Omega u\sin\phi \qquad + \nu\nabla^2 v$$

$U \cdot U/L$ $\quad$ $U \cdot U/a$ $\quad$ $U \cdot W/a$ $\quad$ $\dfrac{\Delta P}{\rho L}$ $\quad$ $U f_0$ $\quad$ $W f_0$ $\quad$ $\dfrac{\nu U}{H^2}$

(Horizontal Momentum Equation)

Scales

$$U \approx 10 \text{ m s}^{-1}$$
$$W \approx 0.01 \text{ m s}^{-1}$$
$$L \approx 10^6 \text{ m}$$
$$H \approx 10^4 \text{ m}$$
$$L/U \approx 10^5 \text{ s}$$

$$\Delta P \approx 1000 \text{ Pa}$$
$$\rho \approx 1 \text{ kg m}^{-3}$$
$$\Delta\rho/\rho \approx 10^{-2}$$
$$f_0 \approx 10^{-4} \text{ s}^{-1}$$

$$a \approx 10^7 \text{ m}$$
$$g \approx 10 \text{ m s}^{-2}$$
$$\nu \approx 10^{-5} \text{ m}^2 \text{ s}^{-1}$$

$$\frac{Du}{Dt} - \frac{uv\tan\phi}{r} + \frac{uw}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2\Omega v \sin\phi - 2\Omega w \cos\phi + \nu\nabla^2 u$$

$$\frac{Dv}{Dt} + \frac{u^2\tan\phi}{r} + \frac{vw}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2\Omega u \sin\phi + \nu\nabla^2 v$$

U·U/L	U·U/a	U·W/a	ΔP/ρL	Uf	Wf	$\nu U/H^2$
10^{-4}	10^{-5}	10^{-8}	10^{-3}	10^{-3}	10^{-6}	10^{-12}

(Horizontal Momentum Equation)

Largest Terms

$$\frac{Du}{Dt} - \frac{uv\tan\phi}{r} + \frac{uw}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2\Omega v\sin\phi - 2\Omega w\cos\phi + \nu\nabla^2 u$$

$$\frac{Dv}{Dt} + \frac{u^2\tan\phi}{r} + \frac{vw}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2\Omega u\sin\phi + \nu\nabla^2 v$$

U·U/L	U·U/a	U·W/a	ΔP/ρL	Uf	Wf	$\nu U/H^2$
10^{-4}	10^{-5}	10^{-8}	10^{-3}	10^{-3}	10^{-6}	10^{-12}

Dominant Balance

Only retain the largest terms…

Pressure Gradient Force

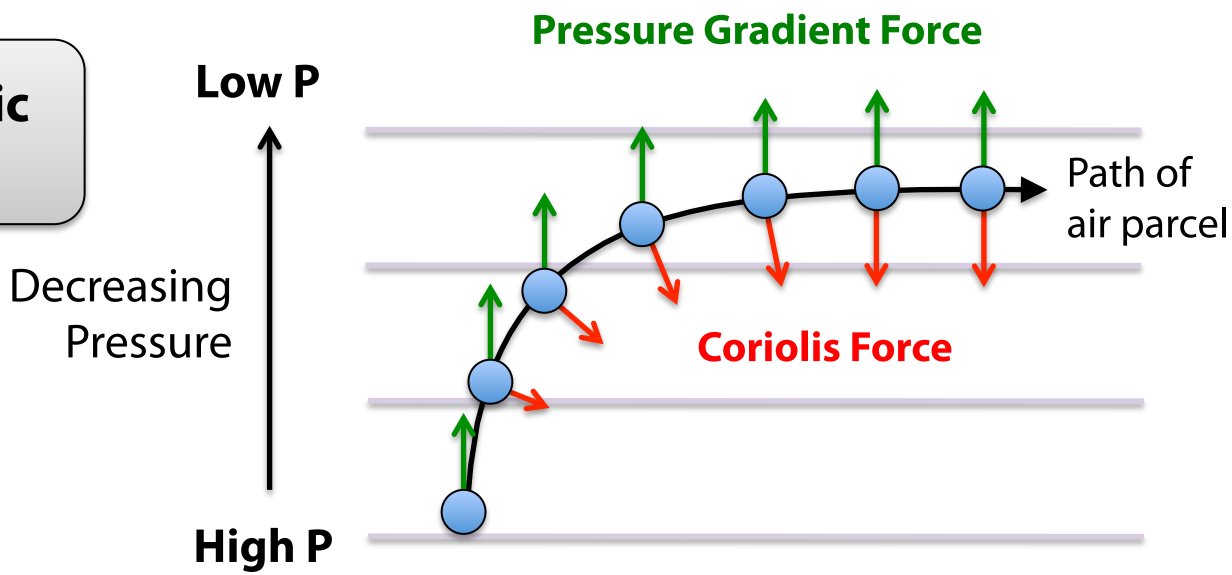
Coriolis Force

$$\frac{1}{\rho}\frac{\partial p}{\partial x} - 2\Omega v \sin\phi = 0$$

$$\frac{1}{\rho}\frac{\partial p}{\partial y} + 2\Omega u \sin\phi = 0$$

Balance: If no time derivative term is present the system is static. The remaining terms must balance each other for the equation to hold.

Geostrophic Balance



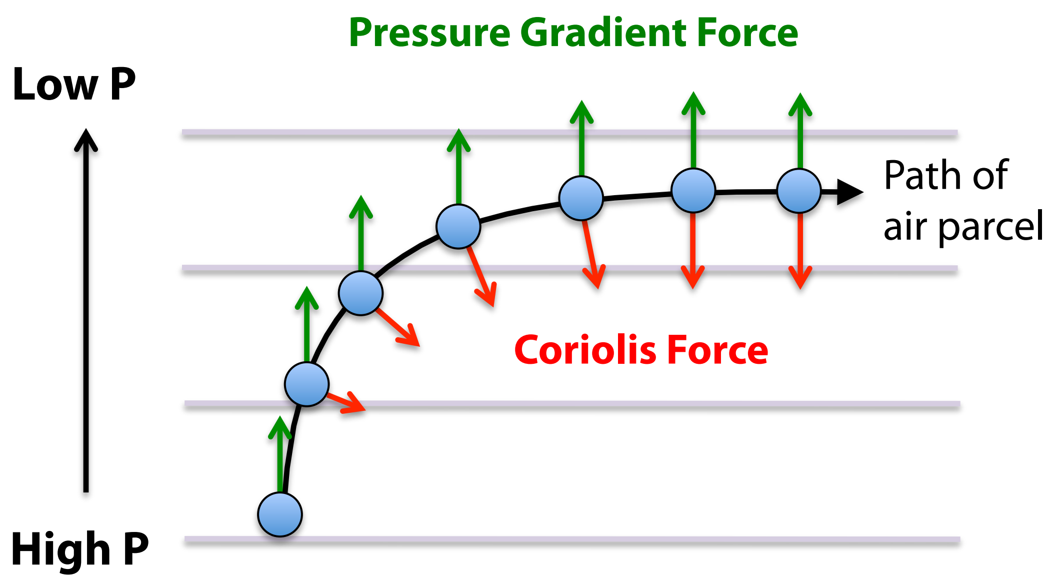
(Horizontal Momentum Equation)

Pressure Gradient Force

Coriolis Force

$$\frac{1}{\rho}\frac{\partial p}{\partial x} - 2\Omega v \sin\phi = 0$$

$$\frac{1}{\rho}\frac{\partial p}{\partial y} + 2\Omega u \sin\phi = 0$$



Definition: For large-scale mid-latitudinal flows there is an intrinsic **balance between pressure gradient force and Coriolis force**. This balance is known as **geostrophic balance** and leads to air parcels traveling along lines of constant pressure.

Scale Analysis (Vertical Momentum)

$$\frac{Dw}{Dt} - \frac{u^2 + v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + 2\Omega u \cos\phi + \nu \nabla^2 w$$

$$W \cdot U/L \qquad U \cdot U/a \qquad \frac{\Delta P}{\rho H} \qquad g \qquad U f_0 \qquad \frac{\nu W}{H^2}$$

(Vertical Momentum Equation)

$$U \approx 10 \text{ m s}^{-1}$$
$$W \approx 0.01 \text{ m s}^{-1}$$
$$L \approx 10^6 \text{ m}$$
$$H \approx 10^4 \text{ m}$$
$$L/U \approx 10^5 \text{ s}$$

Scales

$$\Delta P \approx 1000 \text{ Pa}$$
$$\rho \approx 1 \text{ kg m}^{-3}$$
$$\Delta\rho/\rho \approx 10^{-2}$$
$$f_0 \approx 10^{-4} \text{ s}^{-1}$$

$$a \approx 10^7 \text{ m}$$
$$g \approx 10 \text{ m s}^{-2}$$
$$\nu \approx 10^{-5} \text{ m}^2 \text{ s}^{-1}$$

$$\frac{Dw}{Dt} - \frac{u^2+v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + 2\Omega u \cos\phi + \nu\nabla^2 w$$

$W \cdot U/L$	$U \cdot U/a$	$P_{sfc}/\rho H$	g	Uf	$\nu W/H^2$
10^{-7}	10^{-5}	10	10	10^{-3}	10^{-15}

(Vertical Momentum Equation)

Largest Terms

$$\frac{Dw}{Dt} - \frac{u^2 + v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + 2\Omega u \cos\phi + \nu \nabla^2 w$$

W·U/L	U·U/a	$P_{sfc}/\rho H$	g	Uf	$\nu W/H^2$
10^{-7}	10^{-5}	10	10	10^{-3}	10^{-15}

Hydrostatic Balance

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = -g$$

Scale Analysis

Question: What are the terms in the equations of motion that are most relevant for large-scale mid-latitude dynamics?

The largest terms in the horizontal and vertical momentum equations lead to two types of balance that dominate the observed flow for **large-scale mid-latitudinal** storm systems:

- **Geostrophic Balance** (Pressure gradient and Coriolis)
- **Hydrostatic Balance** (Pressure gradient and gravity)

Aside: Why is "mid-latitudinal" important?

(Vertical Momentum Equation)

Largest Terms

$$\frac{Dw}{Dt} - \frac{u^2 + v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + 2\Omega u \cos\phi + \nu \nabla^2 w$$

$W \cdot U/L$	$U \cdot U/a$	$P_{sfc}/\rho H$	g	Uf	$\nu W/H^2$
10^{-7}	10^{-5}	10	10	10^{-3}	10^{-15}

The vertical acceleration Dw/Dt is 8 orders of magnitude smaller than **hydrostatic balance**. The ability of the vertical momentum equation to estimate w is essentially nonexistent.

Scale Analysis

Scale analysis of the vertical momentum equation revealed that computing vertical velocity using this equation requires taking the difference of two terms which are **8 orders of magnitude larger** than the acceleration!

Even tiny errors in computing the vertical pressure gradient will lead to **large** errors in the vertical velocity.

Motivates the next question…

Question: How can vertical velocity be computed?

Vertical Velocity?

Vertical motion is ***important***: Rising motion leads to clouds and precipitation.

The vertical acceleration Dw/Dt is 8 orders of magnitude smaller than hydrostatic balance.

The ability to use the vertical momentum equation to estimate w is essentially nonexistent.

- Vertical velocity must be "diagnosed" from some balance
- Note that small scales, thunderstorms, tornadoes use very different characteristic scales, so the vertical momentum equation can be employed in this regime.

We will return to this in a moment…