Introduction to Atmospheric Dynamics

Chapter 1

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Part 6: The Dynamical Equations with Vertical Pressure Coordinate

Pressure Coordinates

What do we need:

- An expression for **pressure gradient force**
- Some way to express **the material derivative**
- Some way to express **vertical velocity**
- An expression for the **thermodynamic equation**
- An expression for the **continuity equation**
- An expression representing **hydrostatic balance**

Horizontal Momentum Eq'n

Pressure Coordinates

Horizontal momentum equations, **pressure coordinate**, no viscosity:

$$\frac{Du}{Dt} = -\left(\frac{\partial \Phi}{\partial x}\right)_p + fv$$

$$\frac{Dv}{Dt} = -\left(\frac{\partial \Phi}{\partial y}\right)_p - fu$$

Observe density no longer explicitly present.

$$\frac{D\mathbf{u}}{Dt} + f\mathbf{k} \times \mathbf{u} = -\nabla_p \Phi$$

Note: The subscript p implies that these expressions are evaluated on constant pressure surfaces.

Sometimes the subscript is omitted, but the presence of the geopotential Φ tells you that the expression is in pressure coordinates.

Material Derivative

Height Coordinates

By construction: $$\frac{Dx}{Dt} = u, \frac{Dy}{Dt} = v, \frac{Dz}{Dt} = w$$

Recall that to obtain the material derivative in height coordinates we performed a Taylor expansion of the temperature to obtain:

$$\begin{aligned}\left(\frac{DT}{Dt}\right)_z &= \left(\frac{\partial T}{\partial t}\right)_z + \frac{Dx}{Dt}\left(\frac{\partial T}{\partial x}\right)_z + \frac{Dy}{Dt}\left(\frac{\partial T}{\partial y}\right)_z + \frac{Dz}{Dt}\left(\frac{\partial T}{\partial z}\right)_z \\ &= \left(\frac{\partial T}{\partial t}\right)_z + u\left(\frac{\partial T}{\partial x}\right)_z + v\left(\frac{\partial T}{\partial y}\right)_z + w\left(\frac{\partial T}{\partial z}\right)_z\end{aligned}$$

Vertical Motion

Pressure Coordinates

Definition: The **vertical pressure velocity** is defined as the change in pressure following a fluid parcel.

$$\omega \equiv \frac{Dp}{Dt}$$

This is the velocity of fluid parcels across surfaces of constant pressure.

Material Derivative

Pressure Coordinates

By construction: $$\frac{Dx}{Dt} = u, \frac{Dy}{Dt} = v, \frac{Dp}{Dt} = \omega$$

Performing an analogous expansion to the one used for height coordinates:

$$\begin{aligned}\left(\frac{DT}{Dt}\right)_p &= \left(\frac{\partial T}{\partial t}\right)_p + \frac{Dx}{Dt}\left(\frac{\partial T}{\partial x}\right)_p + \frac{Dy}{Dt}\left(\frac{\partial T}{\partial y}\right)_p + \frac{Dp}{Dt}\left(\frac{\partial T}{\partial p}\right)_p \\ &= \left(\frac{\partial T}{\partial t}\right)_p + u\left(\frac{\partial T}{\partial x}\right)_p + v\left(\frac{\partial T}{\partial y}\right)_p + \omega\left(\frac{\partial T}{\partial p}\right)_p\end{aligned}$$

Vertical Motion

Pressure Coordinates

Question: For upward velocity what is the sign of w? (the vertical velocity in height coordinates)

Question: What is the sign of ω for upward motion?

Recall $\omega = \frac{Dp}{Dt}$

Question: What is the sign of Dp/Dt (change in pressure following the parcel) for upward motion? For downward motion?

Vertical Motion

Pressure Coordinates

Question: What is the sign of Dp/Dt (change in pressure following the parcel) for upward motion? For downward motion?

Something to keep in mind:

- ω is **negative** for **upward** motion
- ω is **positive** for **downward** motion

Thermodynamic Equation

Pressure Coordinates

1 $$c_v \frac{DT}{Dt} + p \frac{D\alpha}{Dt} = J$$

2 $$(c_v + R_d) \frac{DT}{Dt} - \alpha \frac{Dp}{Dt} = J$$

We have derived two equivalent forms of the thermodynamic equation which are valid regardless of vertical coordinate.

$$c_p = c_v + R$$

Material Derivative

$$c_p \left(\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + \omega \frac{\partial T}{\partial p} \right) - \alpha\omega = J$$

$$c_p \left(\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + \omega \frac{\partial T}{\partial p} \right) - \alpha \omega = J$$

Divide by c_p

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + \omega \left(\frac{\partial T}{\partial p} - \frac{\alpha}{c_p} \right) = \frac{J}{c_p}$$

Ideal gas law

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + \omega \left(\frac{\partial T}{\partial p} - \frac{RT}{pc_p} \right) = \frac{J}{c_p}$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + \omega\left(\frac{\partial T}{\partial p} - \frac{RT}{pc_p}\right) = \frac{J}{c_p}$$

Definition: The **static stability parameter** is defined via the relationship

$$S_p = \left(\frac{R_d T}{pc_p} - \frac{\partial T}{\partial p}\right)$$

$$\boxed{\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} - S_p\omega = \frac{J}{c_p}}$$

Static Stability Parameter

Thermodynamic Equation

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} - S_p\omega = \frac{J}{c_p}$$

$$S_p = \left(\frac{R_d T}{p c_p} - \frac{\partial T}{\partial p}\right)$$

If there is no horizontal advection, then the time rate of change of temperature is driven by two factors:

$$\frac{\partial T}{\partial t} = \frac{J}{c_p} + S_p\omega$$

Diabatic heating (radiation, condensation)

Adiabatic rising/ sinking

Continuity Equation

Pressure Coordinates

In height coordinates:

$$\frac{D\rho}{Dt} = -\rho \nabla \cdot \mathbf{u}$$

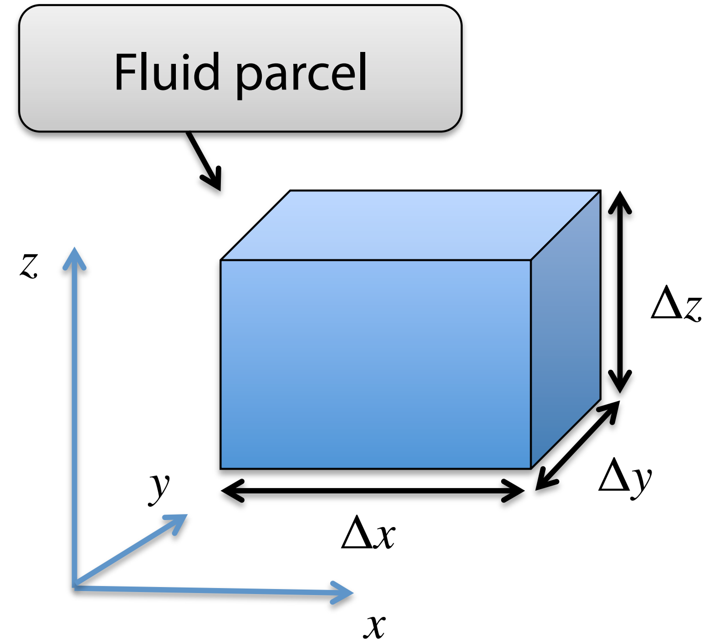
We could try to derive the pressure-coordinate version of the continuity equation from the height-coordinate equation (as we did with the pressure gradient force in the horizontal momentum equations), but…

…it turns out to be easier to re-drive it from the principle of mass conservation.

Start with an air parcel with volume $V = \Delta x \Delta y \Delta z$

Hydrostatic Relation $\frac{\Delta p}{\Delta z} = -\rho g$

$$V = -\Delta x \Delta y \frac{\Delta p}{\rho g}$$



Hence, the mass of this fluid element is:

$$M = \rho V = -\rho \Delta x \Delta y \frac{\Delta p}{\rho g} = -\Delta x \Delta y \Delta p / g$$

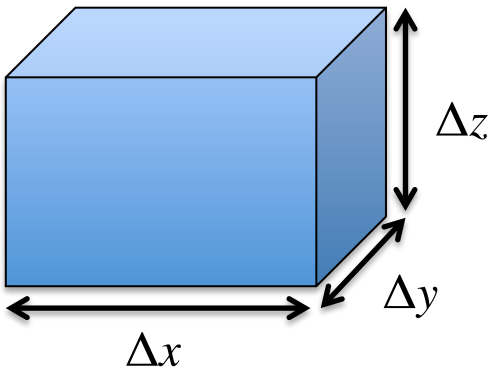
$$\frac{DM}{Dt} = 0$$

Conservation of Mass
(Mass is conserved following a fluid parcel)

$$\frac{1}{M}\frac{DM}{Dt} = 0$$

$$M = -\Delta x \Delta y \Delta p / g$$

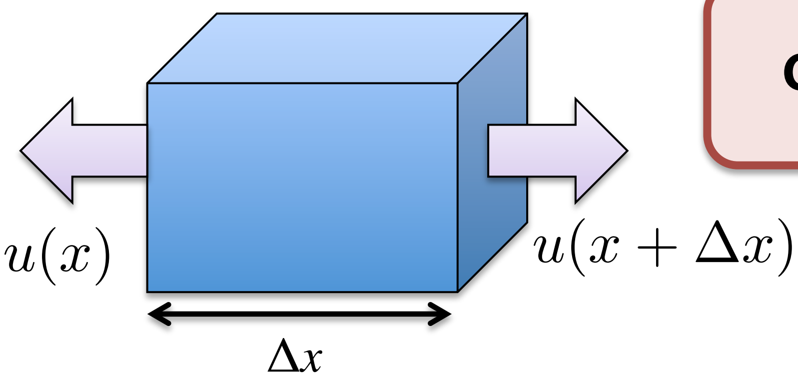
$$\frac{-g}{\Delta x \Delta y \Delta p}\frac{D}{Dt}(\Delta x \Delta y \Delta p)\left(-\frac{1}{g}\right) = 0$$



Product Rule

$$\frac{1}{\Delta x \Delta y \Delta p}\left(\frac{D(\Delta x)}{Dt}\Delta y \Delta p + \frac{D(\Delta y)}{Dt}\Delta x \Delta p + \frac{D(\Delta p)}{Dt}\Delta x \Delta y\right) = 0$$

$$\frac{1}{\Delta x \Delta y \Delta p}\left(\boxed{\frac{D(\Delta x)}{Dt}}\Delta y \Delta p + \frac{D(\Delta y)}{Dt}\Delta x \Delta p + \frac{D(\Delta p)}{Dt}\Delta x \Delta y\right) = 0$$



Question: What is $\frac{D}{Dt}(\Delta x)$?

Change following a fluid parcel

Width of the fluid parcel

How does the width change? Differences in velocity!

$$\frac{D}{Dt}(\Delta x) = u(x + \Delta x) - u(x) = \Delta u$$

$$\frac{1}{\Delta x \Delta y \Delta p}\left(\frac{D(\Delta x)}{Dt}\Delta y \Delta p + \frac{D(\Delta y)}{Dt}\Delta x \Delta p + \frac{D(\Delta p)}{Dt}\Delta x \Delta y\right) = 0$$

$$\Downarrow$$

$$\frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta y} + \frac{\Delta \omega}{\Delta p} = 0$$

Take the Limit

$$\Downarrow$$

Continuity Equation
along pressure surfaces

$$\boxed{\left(\frac{\partial u}{\partial x}\right)_p + \left(\frac{\partial v}{\partial y}\right)_p + \frac{\partial \omega}{\partial p} = 0}$$

Continuity Equation

Pressure Coordinates

$$\left(\frac{\partial u}{\partial x}\right)_p + \left(\frac{\partial v}{\partial y}\right)_p + \frac{\partial \omega}{\partial p} = 0$$

This form of the continuity equation contains **no reference to the density field** and **does not involve time derivatives**.

The **simplicity** of this equation is one of the chief advantages of the isobaric system.

The **ease** of computing the vertical pressure velocity ω is another…

Hydrostatic Equation

Pressure Coordinates

$$\frac{\partial p}{\partial z} = -\rho g$$

Hydrostatic Equation
In height coordinates

$$\frac{\partial z}{\partial p} = -\frac{1}{\rho g}$$

$$\rho = \frac{p}{R_d T}$$

Ideal Gas Law

$$\frac{\partial (gz)}{\partial p} = -\frac{1}{\rho}$$

$$\frac{\partial \Phi}{\partial p} = -\frac{R_d T}{p}$$

Hydrostatic Equation
In pressure coordinates

This equation replaces the vertical momentum equation.

Dynamical Equations

Pressure Coordinates

Momentum Equation

$$\frac{D\mathbf{u}_h}{Dt} + f\mathbf{k} \times \mathbf{u}_h = -\nabla_p \Phi$$

Hydrostatic Relation

$$\frac{\partial \Phi}{\partial p} = -\frac{R_d T}{p}$$

Continuity Equation

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)_p + \frac{\partial \omega}{\partial p} = 0$$

Thermodynamic Equation

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} - S_p \omega = \frac{J}{c_p}$$

Ideal Gas Law

$$p = \rho R_d T$$

Material Derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial p}$$

Pressure Coordinates

In deriving the equations of motion in pressure coordinates, we have:

- Used conservation principles
- Relied heavily on the hydrostatic assumption (note that these equations are not valid if the hydrostatic assumption does not hold).
- Observed that the conservations principles will hold in all coordinate systems.