Introduction to Atmospheric Dynamics
Chapter 1

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Part 5: A First Look at the Vertical Structure of the Atmosphere

Hydrostatic Balance

Although the horizontal atmosphere is in a constant state of motion, vertical velocities are fairly small (especially averaged over the large scale). Consequently, to understand the vertical structure of the atmosphere, we can approximate it to be largely steady.

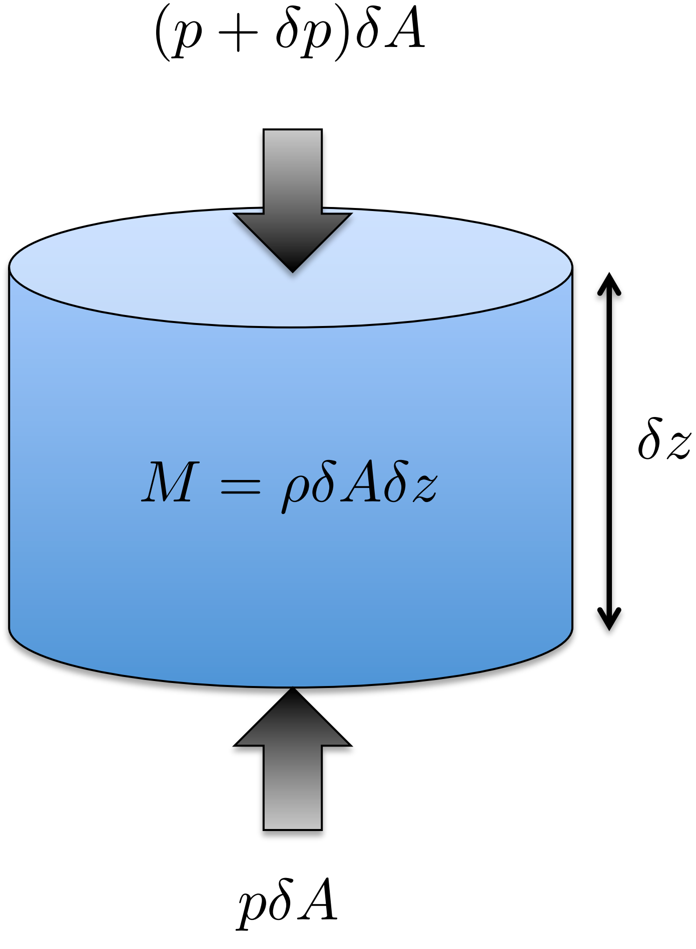
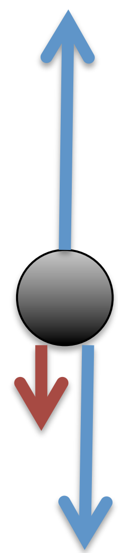


Figure: A vertical column of air of density ρ, horizontal cross-section δA, height δz and mass $M = \varrho\ \delta A\ \delta z$. The pressure at the lower surface is p, the pressure at the upper surface is $p + \delta p$.

Hydrostatic Balance

If the cylinder of air is not accelerating, it must be subject to zero net force. The vertical forces are:



Pressure acting on bottom face

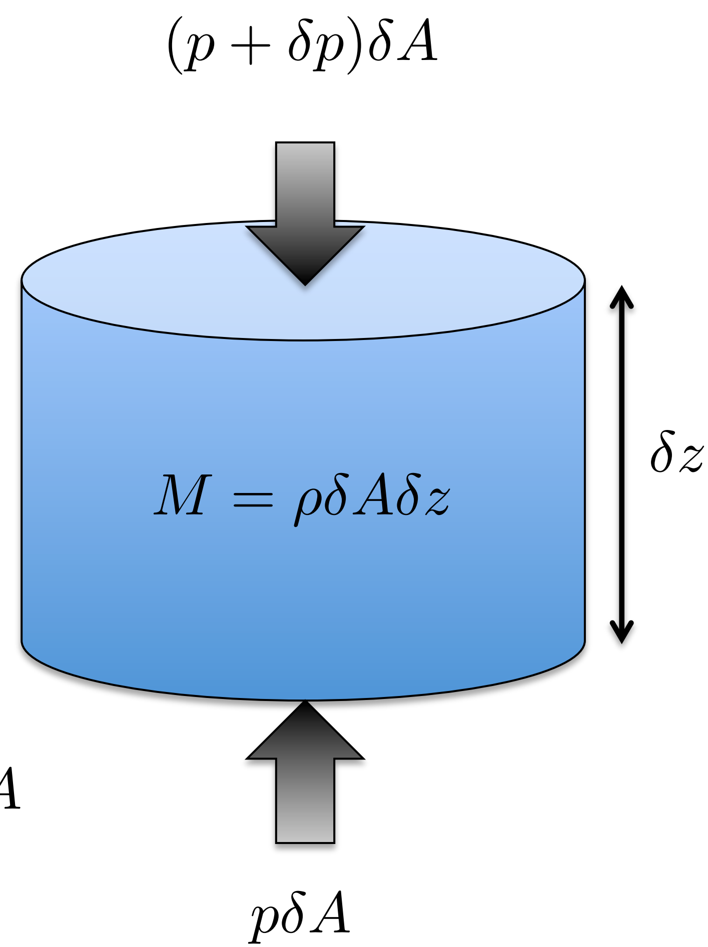
$$F_B = p\delta A$$

Gravity

$$F_g = -gM$$
$$= -g\rho\delta A\delta z$$

Pressure acting on top face

$$F_T = -(p + \delta p)\delta A$$



Hydrostatic Balance

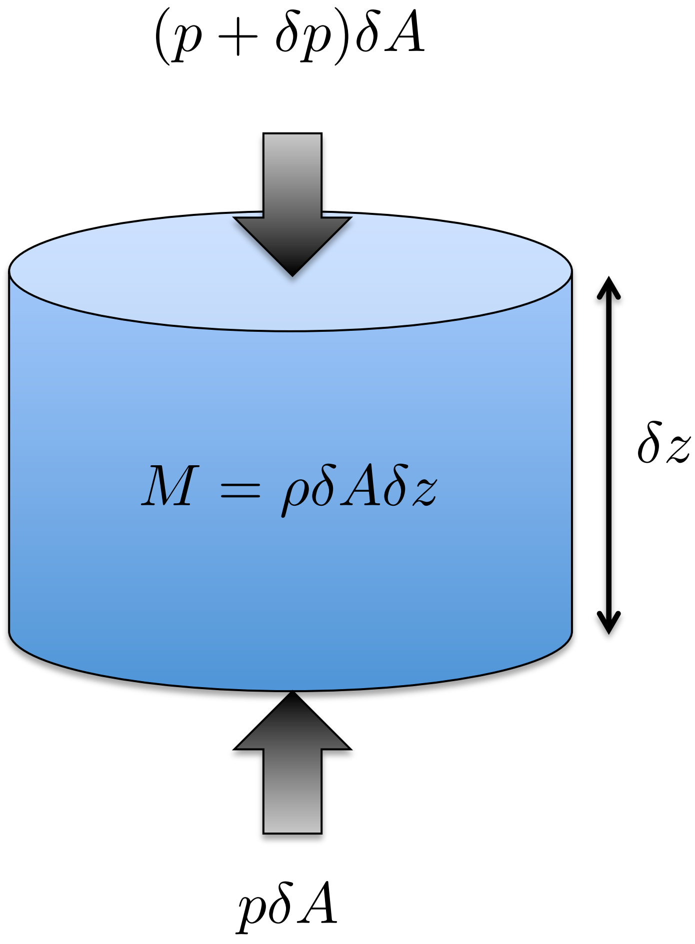
$$F_g + F_T + F_B = 0$$

$$\delta p + g\rho\delta z = 0$$

Taylor Series $\delta p \approx \frac{\partial p}{\partial z}\delta z$

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = -g$$

Hydrostatic Balance



Aside: Consider the special case of an isothermal (constant temperature) atmosphere:

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = -g$$

Hydrostatic Balance

This equation does not give pressure explicitly in terms of height, since the density of air is not known.

Ideal gas law

$$\rho = \frac{p}{R_d T}$$

$$\frac{\partial p}{\partial z} + \frac{pg}{RT} = 0$$

For an isothermal atmosphere $(T = T_0)$ this equation can be exactly solved:

$$p(z) = p_s \exp\left(-\frac{zg}{R_d T}\right)$$

Exponential decay

Aside: Consider the special case of an isothermal (constant temperature) atmosphere:

$$p(z) = p_s \exp\left(-\frac{gz}{R_d T_0}\right)$$

Definition: The **scale height** of an isothermal atmosphere is given by:

$$H = \frac{R_d T_0}{g}$$

$$p(z) = p_s \exp\left(-\frac{z}{H}\right)$$

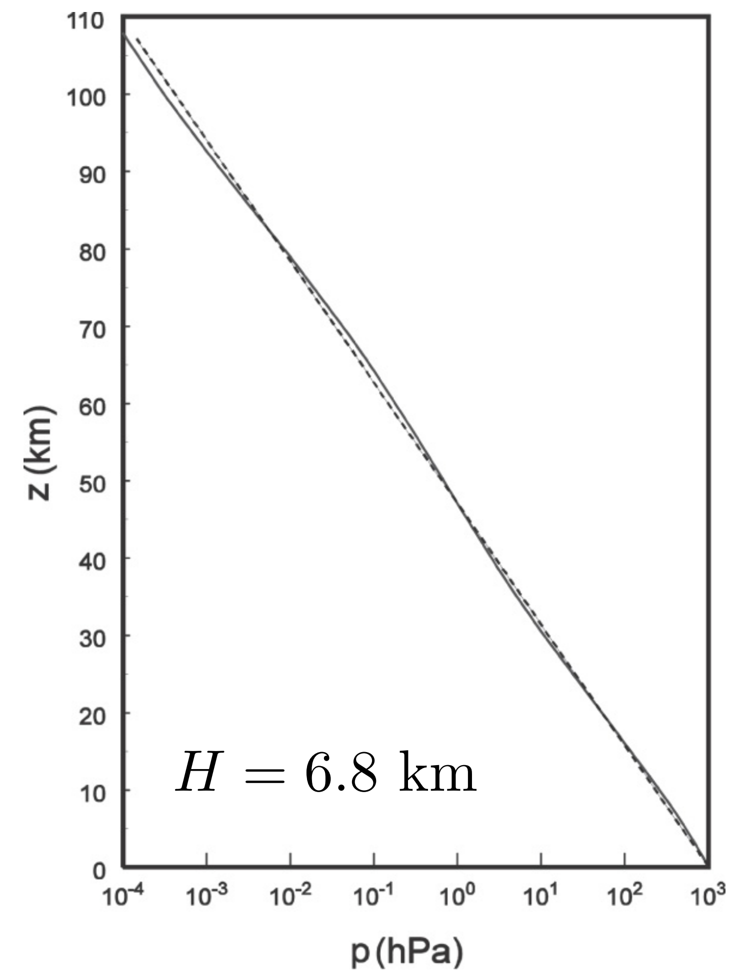
The **scale height** is an example of a quantity which imparts a notion of a "natural measuring stick" for an idealized atmosphere. This notion will generalize to more realistic atmospheric flows as well.

Aside: Consider the special case of an isothermal (constant temperature) atmosphere:

For an isothermal atmosphere $T = T_0$

$$p(z) = p_s \exp\left(-\frac{z}{H}\right)$$

Figure: Observed profile of pressure (solid) plotted against isothermal profile. Observed temperature variations only lead to small variations in the pressure from an exponential profile.



What can we say in the general case?

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = -g$$

Hydrostatic Balance

Integrating from some height z to the top of the atmosphere (z=∞), and noting that p(z=∞)=0 then gives

$$p(z) = g\int_z^\infty \rho dz$$

Mass above height z

Hence the pressure at a given height level is proportional to the total mass of the atmosphere above it.

At the surface, this means $p_s = \frac{gM_a}{4\pi a^2}$ (M_a = total mass of atmosphere)

Geopotential

Recall: Gravity induces a downward force (which we approximate as constant throughout the atmosphere).

$$\frac{\mathbf{F}_g}{m} = -g\mathbf{k}$$

Total gravitational force acting on a fluid parcel

Definition: Geopotential Φ is the "potential function" for gravity. That is, its gradient is equal to the constant of gravity.

$$\nabla\Phi = g\mathbf{k}$$

Geopotential

$$\boxed{\nabla \Phi = g\mathbf{k}}$$

In height coordinates, geopotential is purely a function of z

$$\Rightarrow \quad \frac{d\Phi}{dz} = g$$

Integrate $\Rightarrow$ $$\Phi(z) - \Phi(0) = \int_0^z g dz$$

Define $\Phi(0) = 0$ $\Rightarrow$ $$\Phi(z) = \int_0^z g dz = gz$$

Geopotential

$$\Phi(z) = \int_0^z g dz = gz$$

Recall: Potential energy of an object is given by $\mathrm{PE} = mgh$ where m is the mass and h is the height above a reference point.

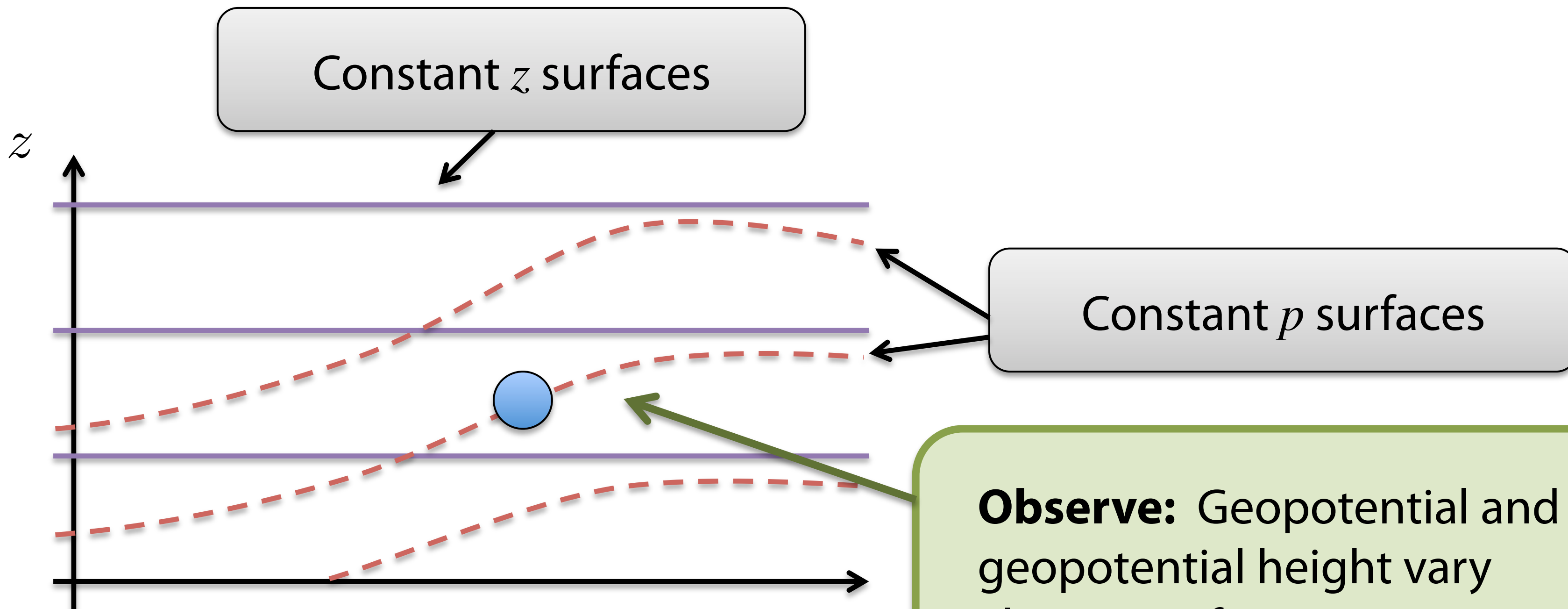
For a fluid, z denotes the height of the fluid parcel. So h and z are the same.

Observe: Geopotential can also be defined as the **potential energy per unit mass** of a fluid parcel lifted to some height z.

$$\frac{\mathrm{PE}}{m} = gz = \Phi$$

Definition: Geopotential height Z is the geopotential divided by gravity. The notion of geopotential height is used when height-based vertical coordinates are not.

$$Z = \frac{\Phi}{g}$$



Observe: Geopotential and geopotential height vary along p surfaces.

The use of **geopotential on constant pressure surfaces** is analogous to the use of **pressure on constant height surfaces**.

Question: How are geopotential and pressure connected?

From $$\frac{d\Phi}{dz} = g \quad \Rightarrow \quad g\,dz = d\Phi$$

From $$\frac{dp}{dz} = -\rho g \quad \Rightarrow \quad g\,dz = -\frac{dp}{\rho}$$

Hydrostatic balance

Ideal gas law $$\rho = \frac{p}{R_d T}$$

$$d\Phi = -\frac{R_d T\,dp}{p}$$

$$d\Phi = -\frac{R_d T dp}{p}$$

Recall: From elementary calculus, $\dfrac{dp}{p} = d\ln p$

$$d\Phi = -R_d T d\ln p$$

Integrate over a layer

$$\Phi(z_2) - \Phi(z_1) = -R_d \int_{p_1}^{p_2} T d\ln p$$

Use geopotential height

$$Z_2 - Z_1 = -\frac{R_d}{g} \int_{p_1}^{p_2} T d\ln p$$

$Z_2 - Z_1$ is the thickness of the layer bounded above by p_2 and below by p_1
This thickness is proportional to the temperature of the layer.

Hypsometric Equation

From hydrostatic balance and the ideal gas law we have

$$Z_2 - Z_1 = -\frac{R_d}{g} \int_{p_1}^{p_2} T \, d \ln p$$

If the temperature in a layer is constant then

$$h = Z_2 - Z_1 = \frac{R_d T}{g} \ln \left(\frac{p_1}{p_2} \right)$$

Hypsometric Equation

This is the relationship between layer thickness and temperature.

Geopotential

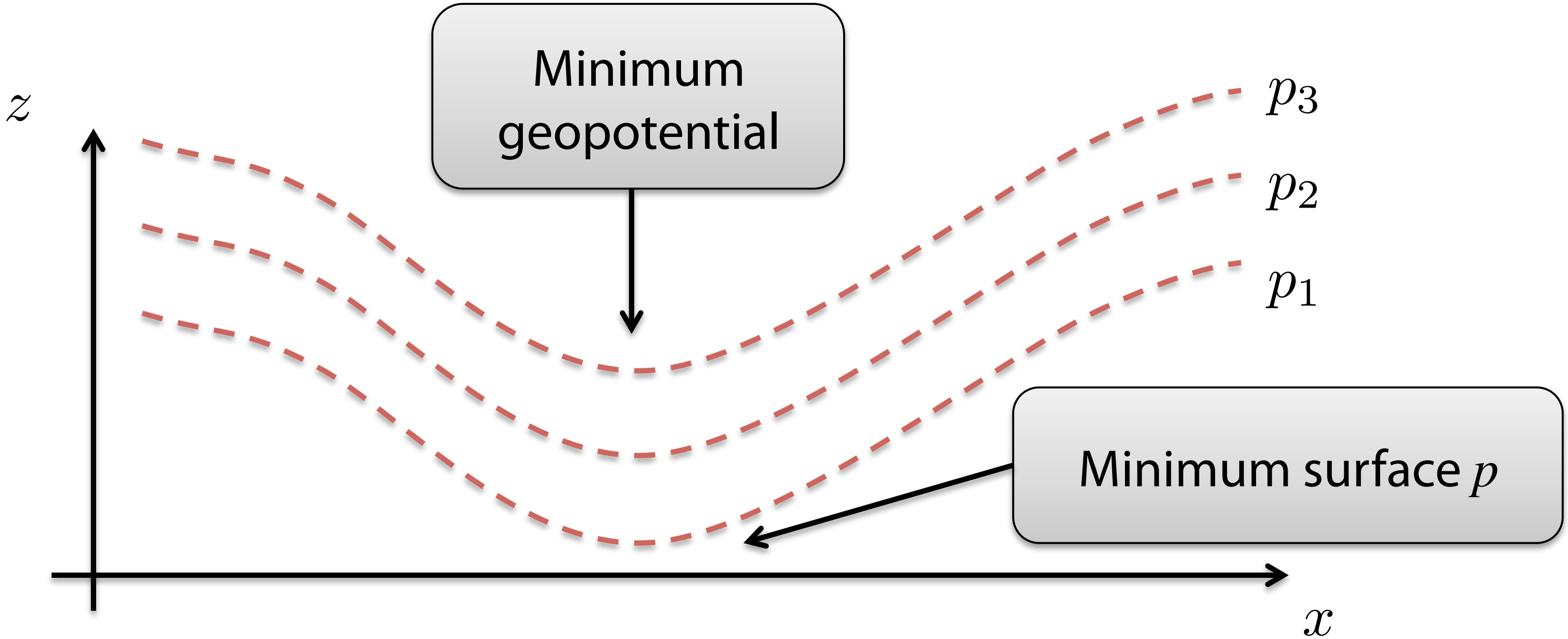


Figure: Geometry of pressure surfaces ($p_1 > p_2 > p_3$) in the vicinity of a pressure minimum.

Observe the minimum surface pressure corresponds to minimum geopotential aloft and vice versa.

Horizontal Momentum

Surface

Body

Apparent

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u} - g\mathbf{k} + fv\mathbf{i} - fu\mathbf{j}$$

Acceleration (change in momentum)

Friction / Viscosity: Opposes motion

Coriolis: Modifies motion

Pressure Gradient Force: Initiates motion

Gravity: Stratification / buoyancy

Vertical Coordinates

Question: Why are we interested in alternative vertical coordinates?

From Holton, p2:

"The general set of … equations governing the motion of the atmosphere is extremely complex; no general solutions are known to exist. …it is necessary to develop models based on systematic simplification of the fundamental governing equations."

Recall: Two goals of dynamic meteorology:

1. Understand atmospheric motions (diagnosis)
2. Predict future atmospheric motions (prognosis)

Use of alternative vertical coordinates ***simplifies*** the equations of motion.

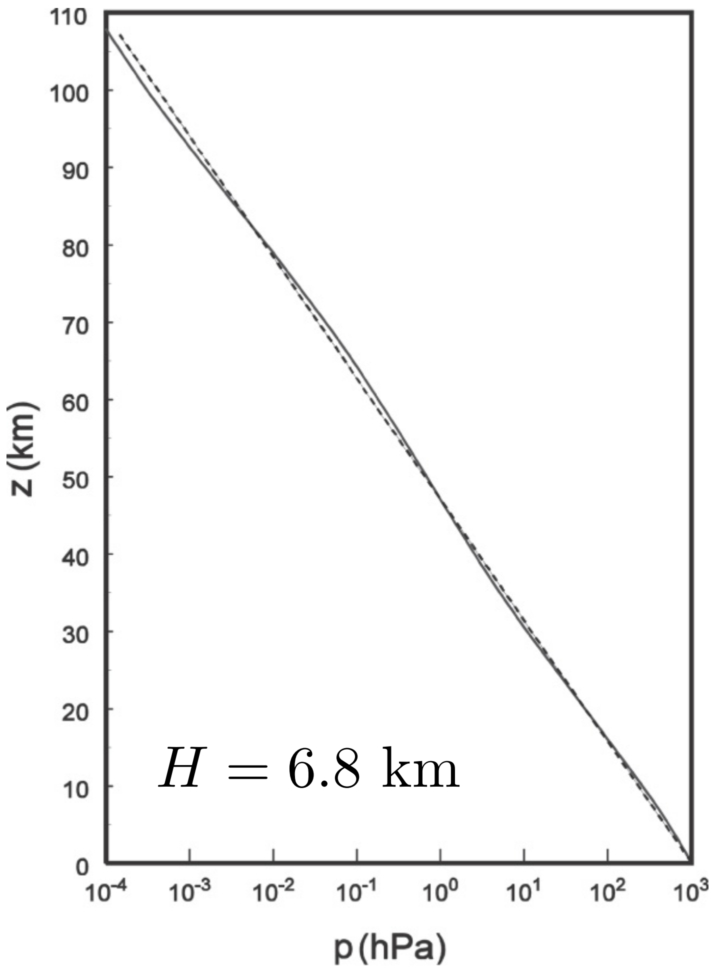
Pressure Coordinates

Recall: For an atmosphere in hydrostatic balance, pressure corresponds to the total mass of the atmosphere above that point.

- Consequently, pressure must decrease monotonically with height.

- This makes pressure a good choice as a vertical coordinate.

We shall see that the use of pressure as a vertical coordinate greatly ***simplifies*** the equations of motion.

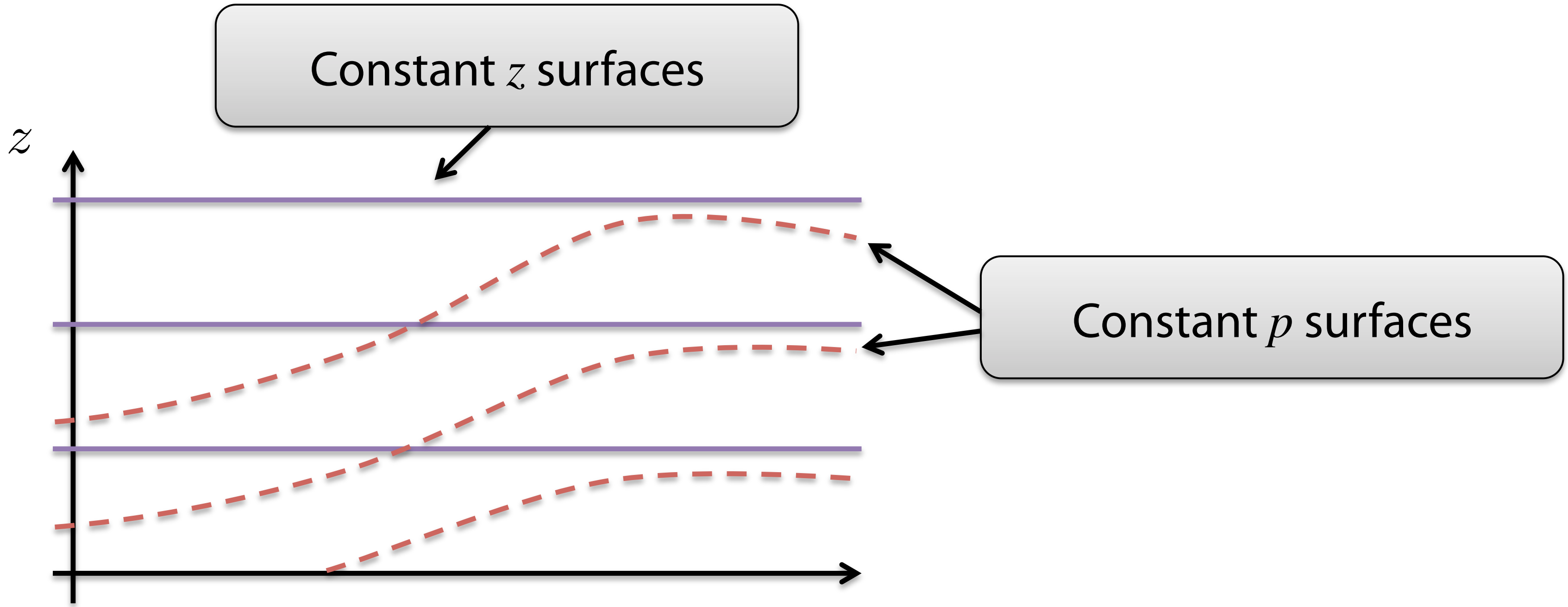


Pressure Coordinates

What do we need:

- An expression for **pressure gradient force**
- Some way to express **derivatives** (gradient, material derivative)
- Some way to express **vertical velocity**
- An expression for the continuity equation

Question: What is the analogue to a pressure gradient in pressure coordinates?

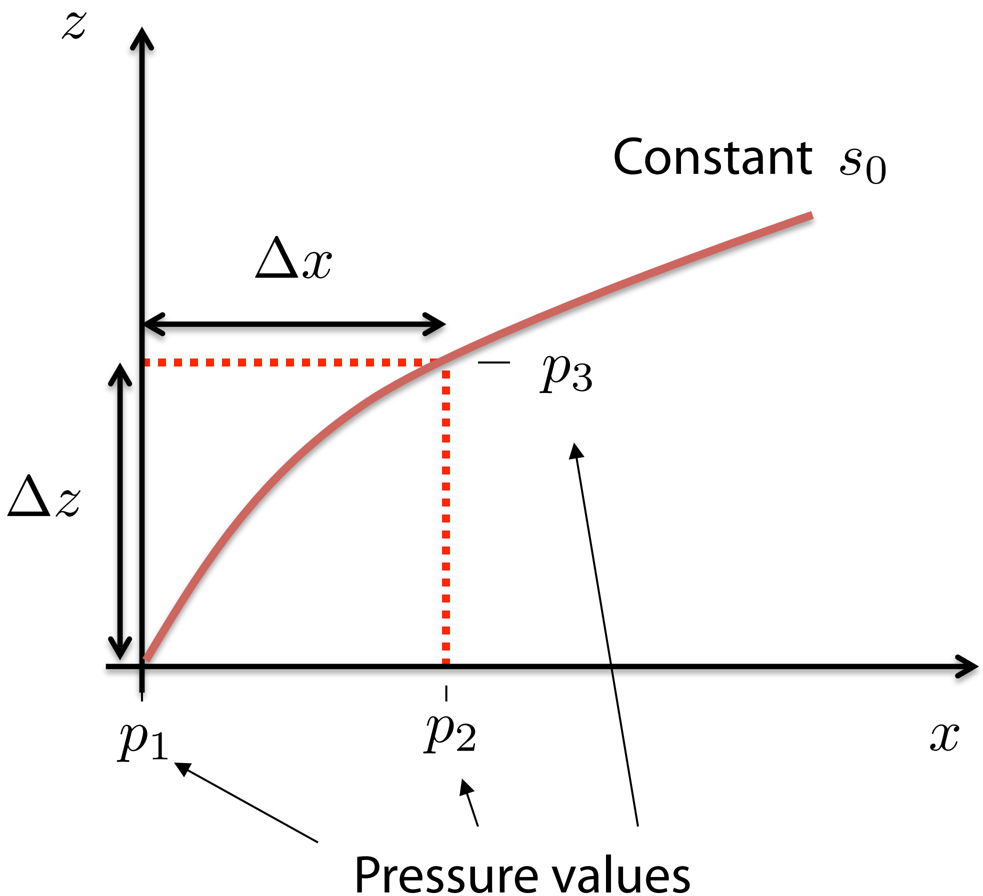


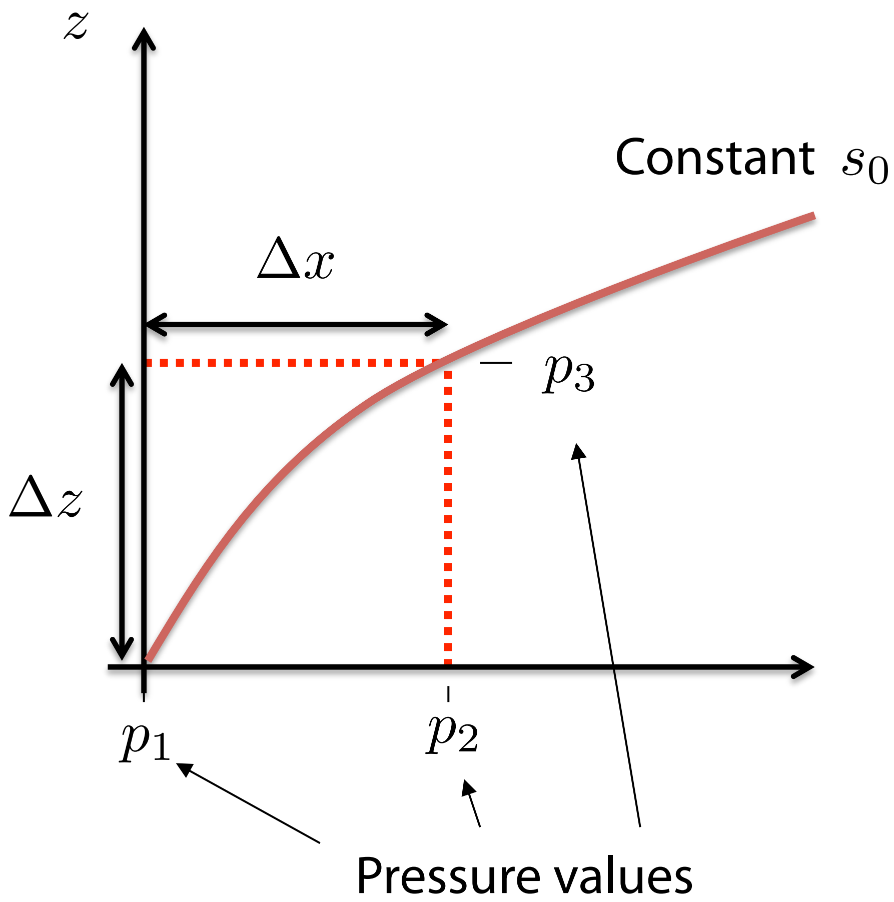
Observe: Pressure is constant along p surfaces, so it will always be zero gradient along these surfaces. If fluid parcels are traveling along p surfaces there must be some analogue to pressure gradient force.

Generalized Vertical Coords.

Any expression s that is a **single-valued monotonic** function of height with $\partial s / \partial z \neq 0$ can also be used as a vertical coordinate.

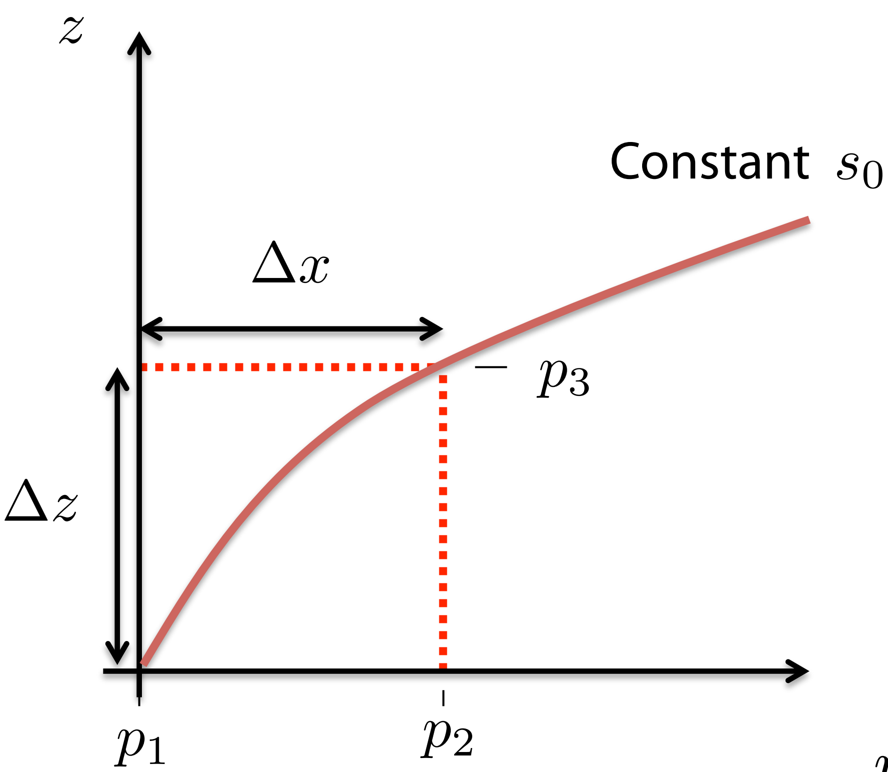
However, since velocity is represented along surfaces of constant s, the notion of a **pressure gradient** along s must be defined.





Change of pressure along s:

$$\frac{p_3 - p_1}{\Delta x}$$



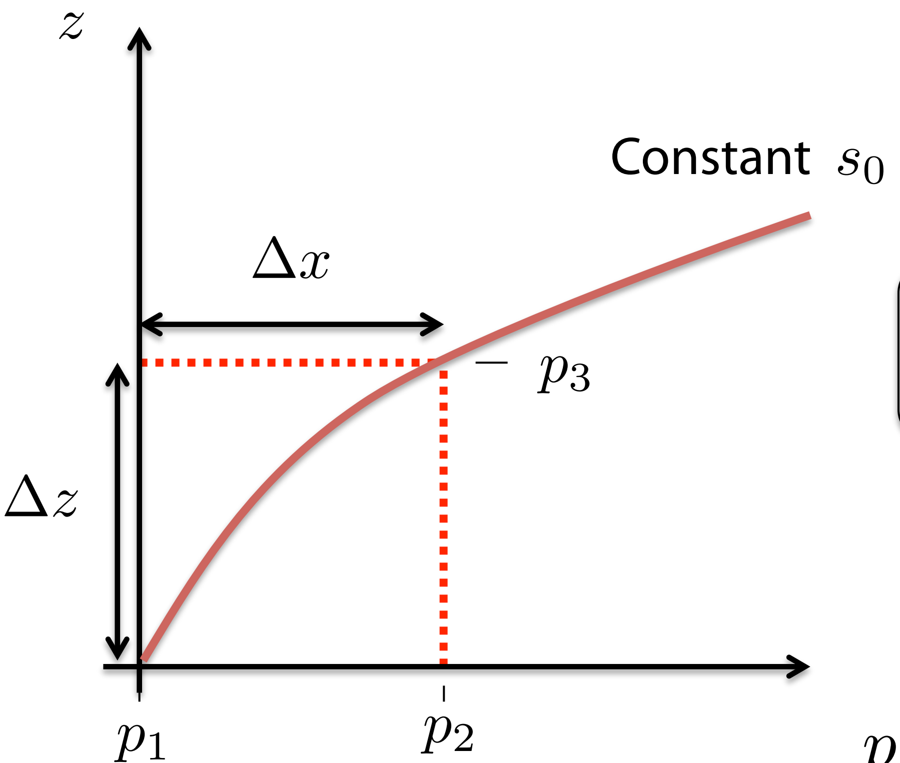
Change of pressure along s:

$$\frac{p_3 - p_1}{\Delta x}$$

Write in terms of p_2 :

$$\frac{p_3 - p_1}{\Delta x} = \frac{p_3 - p_2}{\Delta x} + \frac{p_2 - p_1}{\Delta x}$$

$$\Rightarrow \frac{p_3 - p_1}{\Delta x} = \frac{p_3 - p_2}{\Delta z}\frac{\Delta z}{\Delta x} + \frac{p_2 - p_1}{\Delta x}$$



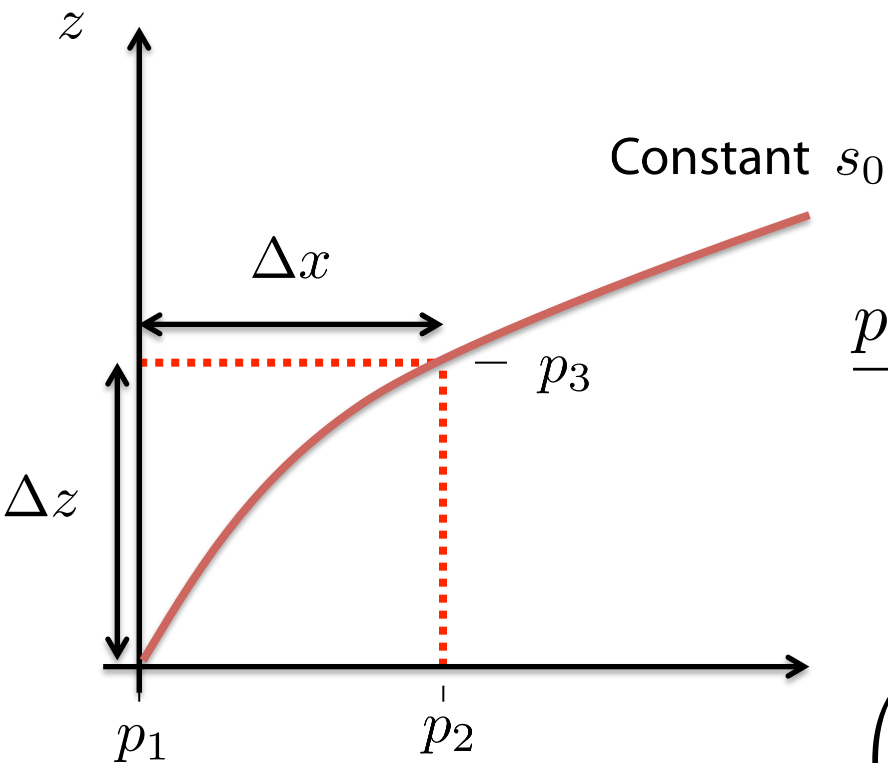
Change in pressure along z with x held fixed

Change in height when moving along s = const

$$\frac{p_3 - p_1}{\Delta x} = \frac{p_3 - p_2}{\Delta z}\frac{\Delta z}{\Delta x} + \frac{p_2 - p_1}{\Delta x}$$

Change in pressure along $s = s_0$

Change in pressure along x with z held fixed



$$\frac{p_3 - p_1}{\Delta x} = \frac{p_3 - p_2}{\Delta z}\frac{\Delta z}{\Delta x} + \frac{p_2 - p_1}{\Delta x}$$

$$\Delta x \to 0 \quad \Delta z \to 0$$

$$\left(\frac{\partial p}{\partial x}\right)_s = \frac{\partial p}{\partial z}\left(\frac{\partial z}{\partial x}\right)_s + \left(\frac{\partial p}{\partial x}\right)_z$$

Chain Rule

$$\left(\frac{\partial p}{\partial x}\right)_s = \frac{\partial s}{\partial z}\left(\frac{\partial z}{\partial x}\right)_s \frac{\partial p}{\partial s} + \left(\frac{\partial p}{\partial x}\right)_z$$

$$\left(\frac{\partial p}{\partial x}\right)_s = \frac{\partial s}{\partial z}\left(\frac{\partial z}{\partial x}\right)_s \frac{\partial p}{\partial s} + \left(\frac{\partial p}{\partial x}\right)_z$$

This is the expression for the horizontal pressure gradient for **ANY** choice of vertical coordinate s.

Plug in $s = p$ (pressure)

$$\underbrace{\cancel{\left(\frac{\partial p}{\partial x}\right)_p}}_{0} = \frac{\partial p}{\partial z}\left(\frac{\partial z}{\partial x}\right)_p \underbrace{\cancel{\frac{\partial p}{\partial p}}}_{1} + \left(\frac{\partial p}{\partial x}\right)_z \quad \Rightarrow \quad \left(\frac{\partial p}{\partial x}\right)_z = -\frac{\partial p}{\partial z}\left(\frac{\partial z}{\partial x}\right)_p$$

Pressure gradient on p surfaces

$$\left(\frac{\partial p}{\partial x}\right)_z = -\frac{\partial p}{\partial z}\left(\frac{\partial z}{\partial x}\right)_p$$

Hydrostatic Relation

$$\frac{\partial p}{\partial z} = -\rho g$$

Definition of geopotential

$$\frac{\partial z}{\partial x} = \frac{1}{g}\frac{\partial (gz)}{\partial x} = \frac{1}{g}\frac{\partial \Phi}{\partial x}$$

$$\boxed{\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_z = \left(\frac{\partial \Phi}{\partial x}\right)_p}$$

The pressure gradient force simply reduces to the gradient of the geopotential on constant pressure surfaces!

Horizontal momentum equations, **height coordinate**, no viscosity:

$$\frac{Du}{Dt} = -\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_z + fv$$

$$\frac{Dv}{Dt} = -\frac{1}{\rho}\left(\frac{\partial p}{\partial y}\right)_z - fu$$

$$\boxed{\frac{D\mathbf{u}}{Dt} + f\mathbf{k} \times \mathbf{u} = -\frac{1}{\rho}\nabla_z p}$$

Horizontal momentum equations, **pressure coordinate**, no viscosity:

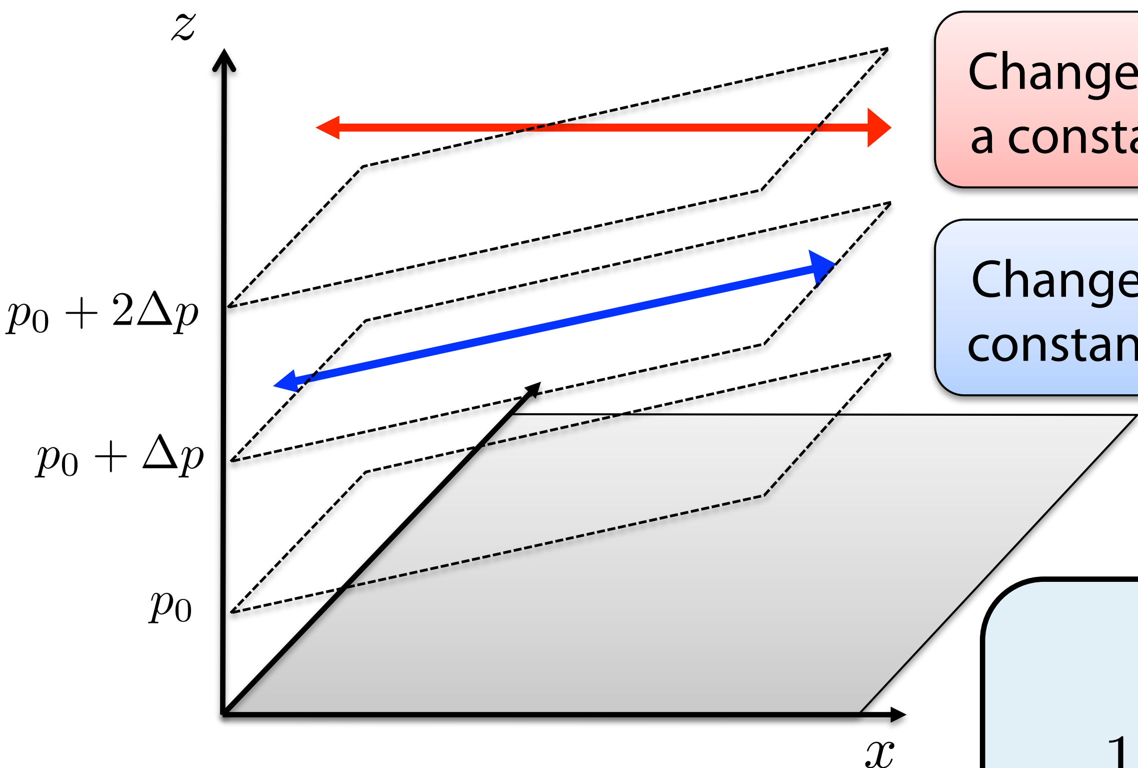
$$\frac{Du}{Dt} = -\left(\frac{\partial \Phi}{\partial x}\right)_p + fv$$

$$\frac{Dv}{Dt} = -\left(\frac{\partial \Phi}{\partial y}\right)_p - fu$$

Observe density no longer explicitly present.

$$\boxed{\frac{D\mathbf{u}}{Dt} + f\mathbf{k} \times \mathbf{u} = -\nabla_p \Phi}$$

Question: What is the force that **initiates** motion?



Change in pressure along a constant height surface

Change in height along a constant pressure surface

Pressure Gradient Force

$$-\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_z = -\left(\frac{\partial \Phi}{\partial x}\right)_p$$