Introduction to Atmospheric Dynamics

Chapter 1

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Part 3: The Continuity Equation

Question: What are the basic physical principles that govern the atmosphere?

Conservation of Mass: Mass is not created nor destroyed. The total mass of the atmosphere is (to a very close approximation) constant over time periods of interest.

Conservation of Mass

Conservation of Mass: Mass is not created nor destroyed. The total mass of the atmosphere is (to a very close approximation) constant over time periods of interest.

Continuity: The fluid is continuous (it contains no holes). This is a fundamental assumption underlying atmospheric motions.

The **Continuity Equation** is used to describe conservation of atmospheric mass.

Conservation

- There are certain parameters (energy, momentum, mass, air, water, ozone, number of atoms, etc.) that must be ***conserved***.

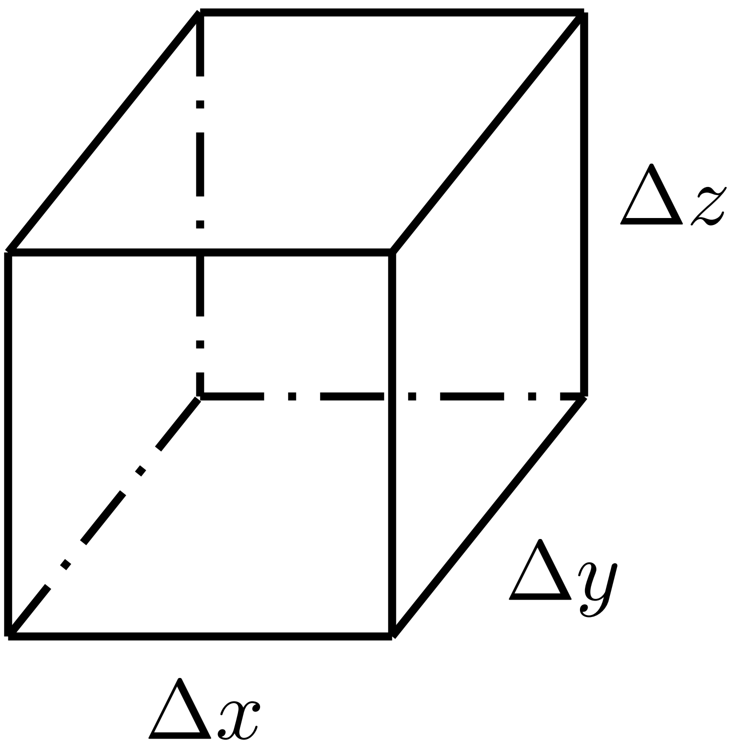
- ***Conservation*** means that in an isolated system that parameter remains constant. It's not created. It's not destroyed. But it can move around.

- Is the Earth's atmosphere isolated?

The Earth's atmosphere is not exactly isolated (escape to space, escape beneath the surface), but to a close approximation this is the case!

The Continuity Equation

Some region of the atmosphere
(not a fluid parcel)



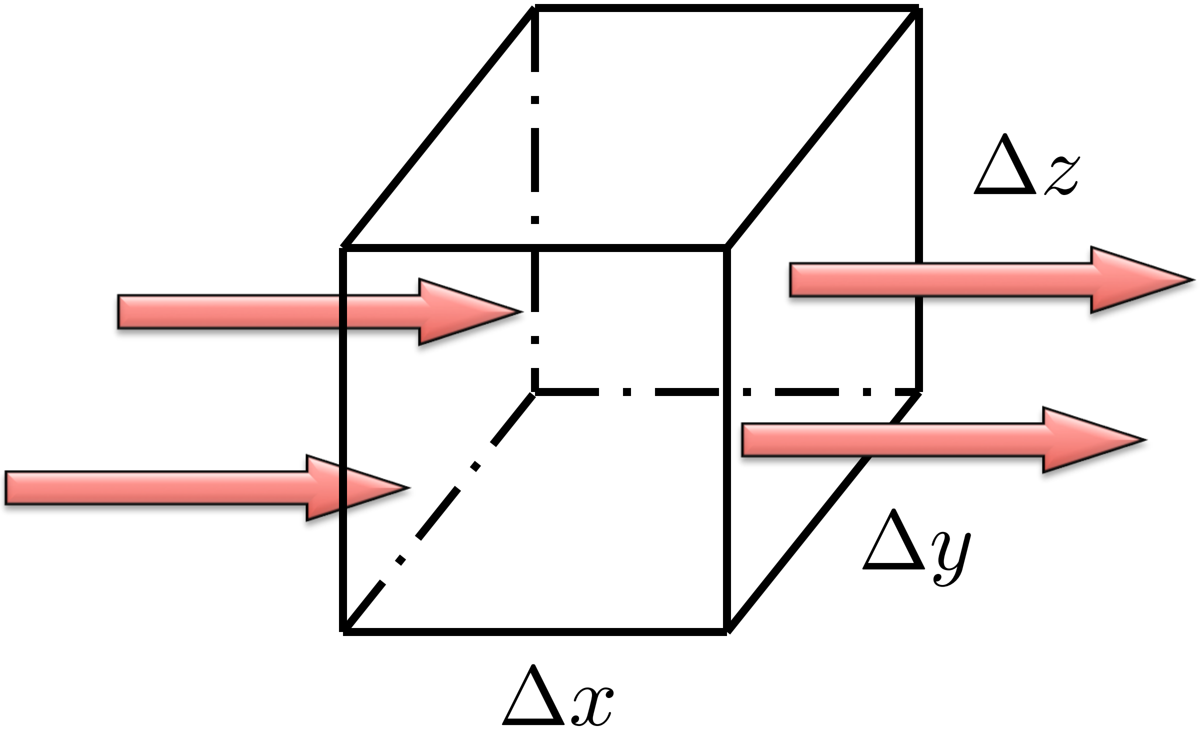
Variables we will need

ρ Density (mass per unit volume)

$V = \Delta x \Delta y \Delta z$ Volume

$m = \rho V$ Mass

In the **Eulerian frame** we have a fixed volume and the fluid flows through it.



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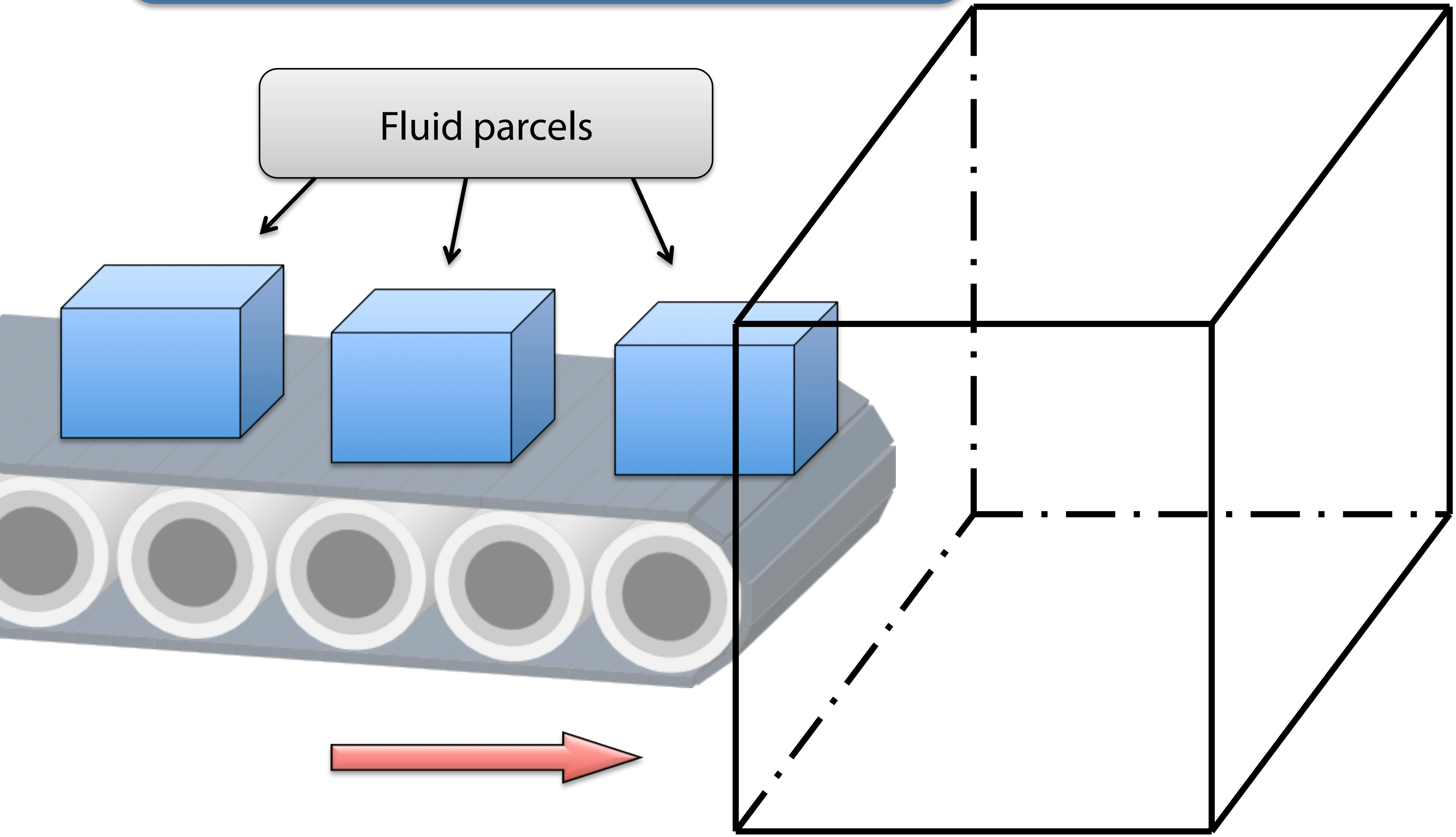
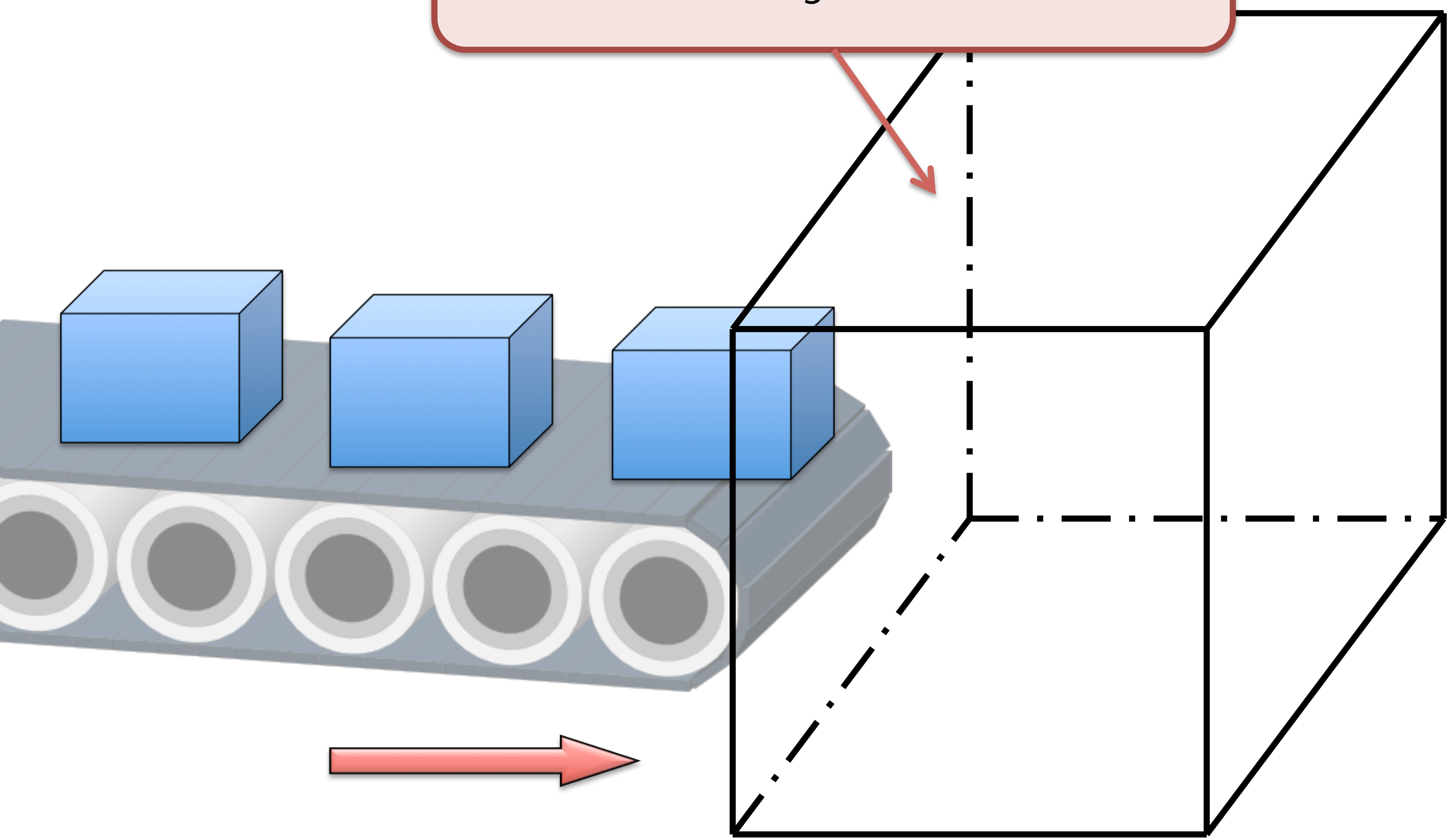
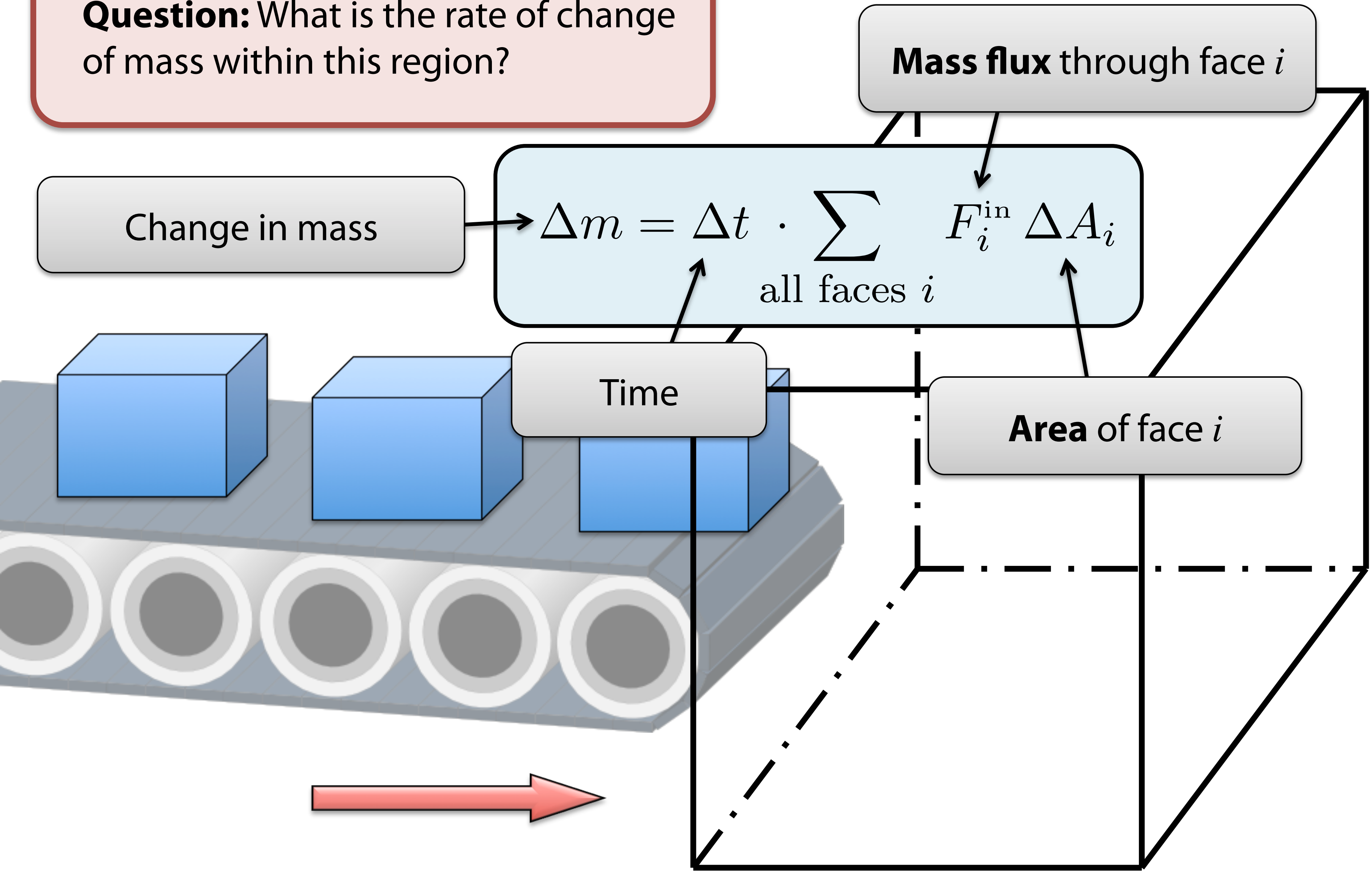


Question: What is the rate of change of mass within this region?



Question: What is the rate of change of mass within this region?

$$\Delta m = \Delta t \cdot \sum_{\text{all faces } i} F_i^{\text{in}} \, \Delta A_i$$



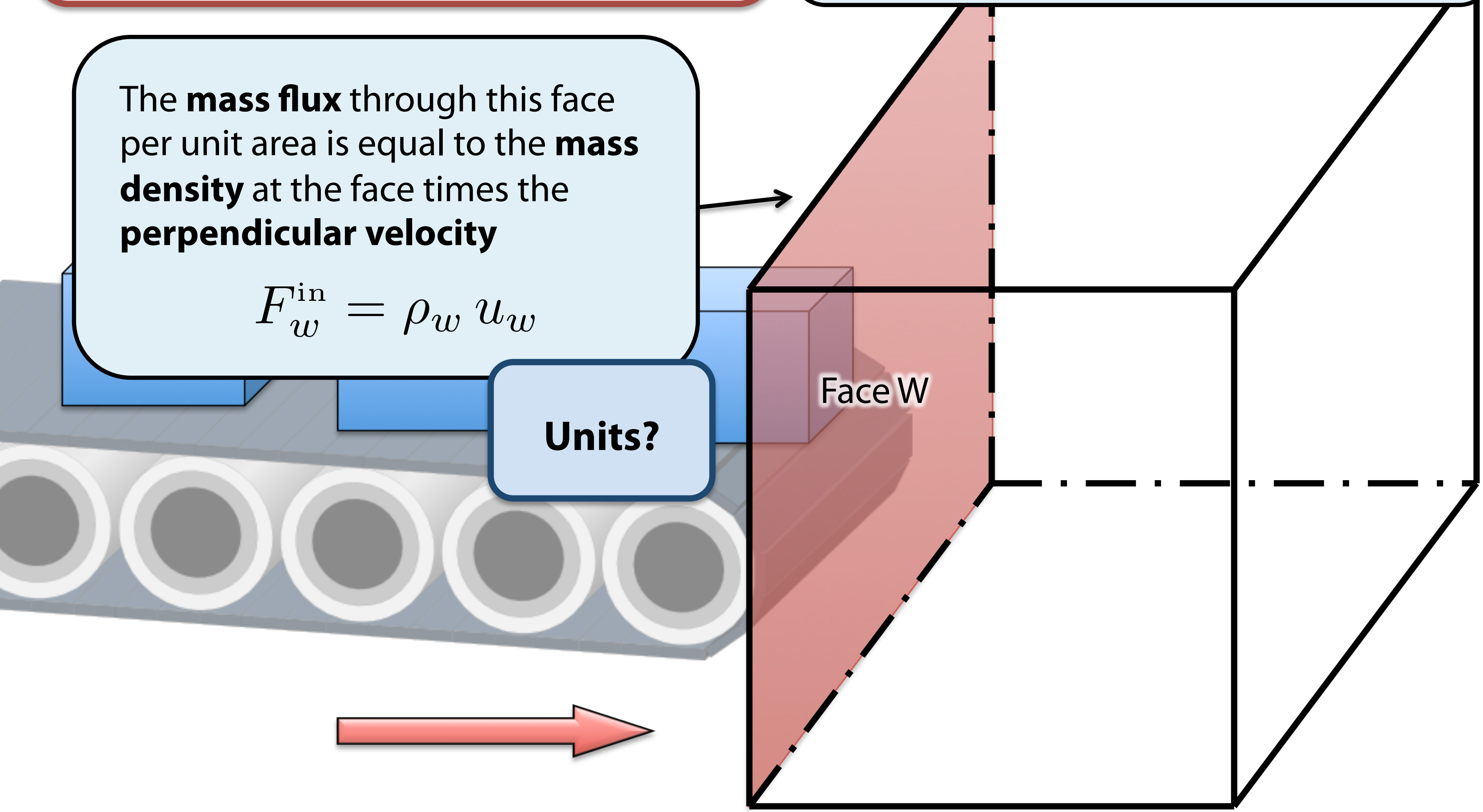
Question: What is the rate of change of mass within this region?

$$\Delta m = \Delta t \cdot \sum_{\text{all faces } i} F_i^{\text{in}} \, \Delta A_i$$

The **mass flux** through this face per unit area is equal to the **mass density** at the face times the **perpendicular velocity**

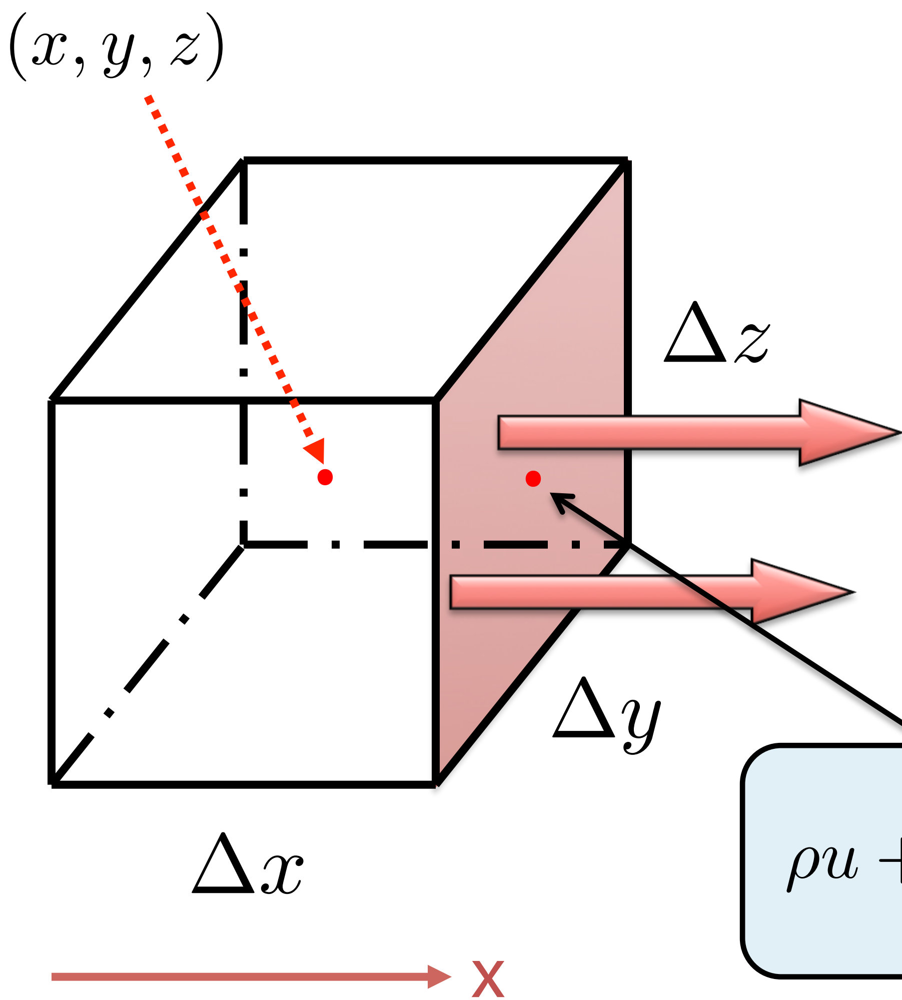
$$F_w^{\text{in}} = \rho_w \, u_w$$

Units?



Definition: The rate of flow of fluid per unit area is the **mass flux**.

$$\mathbf{F} = \rho \mathbf{u}$$



ρu Mass flux at (x, y, z) in the x direction

$$\frac{kg}{m^3} \cdot \frac{m}{s} = \frac{kg}{s} \cdot m^{-2}$$

Mass flux is mass per unit time per unit area

$$\rho u + \left(\frac{\partial(\rho u)}{\partial x} \right) \frac{\Delta x}{2} + \cancel{h.o.t.}$$

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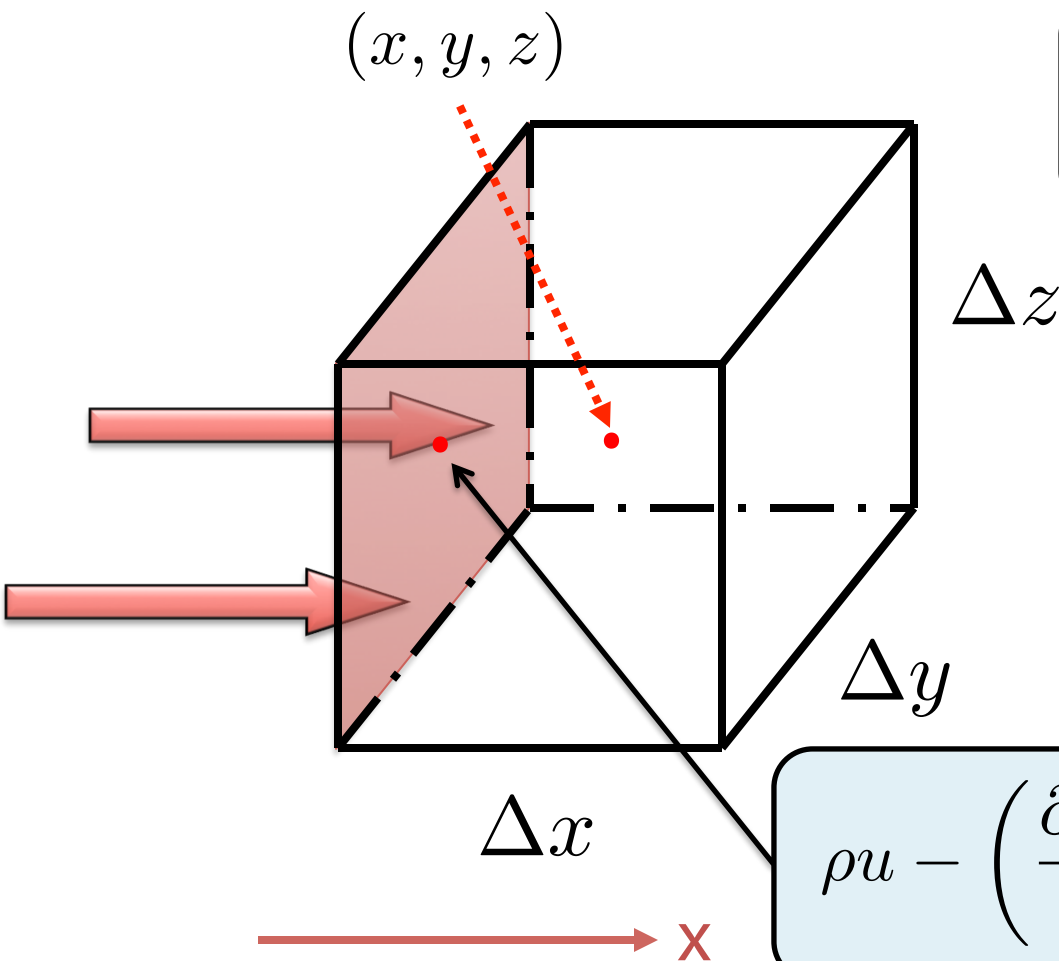


ρu Mass flux at (x, y, z) in the x direction

$$\frac{kg}{m^3} \cdot \frac{m}{s} = \frac{kg}{s} \cdot m^{-2}$$

Mass flux is mass per unit time per unit area

$$\rho u - \left(\frac{\partial (\rho u)}{\partial x} \right) \frac{\Delta x}{2} + \cancel{h.o.t.}$$

Question: What is the rate of change of mass within this region?

The change of mass in the box is equal to the mass that flows into the box minus the mass that flows out of the box.

$$\boxed{\langle Flux \rangle \times \langle Area \rangle}$$

Mass out east (downstream) face

$$F_e \Delta A_e = \left[\rho u + \left(\frac{\partial(\rho u)}{\partial x} \right) \frac{\Delta x}{2} \right] \Delta y \Delta z$$

Mass in west (upstream) face

$$F_w \Delta A_w = \left[\rho u - \left(\frac{\partial(\rho u)}{\partial x} \right) \frac{\Delta x}{2} \right] \Delta y \Delta z$$

Volume of box

Total change in mass per unit time

$$\boxed{\frac{\Delta m}{\Delta t} = F_w \Delta A_w - F_e \Delta A_e = -\frac{\partial(\rho u)}{\partial x} \Delta x \Delta y \Delta z}$$

Question: What is the rate of change of mass within this region?

Extend to three dimensions: The change in mass in the box is equal to the mass that flows into the box minus the mass that flows out of the box.

$$\frac{\Delta m}{\Delta t} = \sum_{\text{all faces } i} F_i^{\text{in}} \Delta A_i = -\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right] V$$

Limit as $\Delta t \to 0$

$$\Delta \rho = \frac{\Delta m}{V}$$

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{u})$$

Continuity Equation (Eulerian)

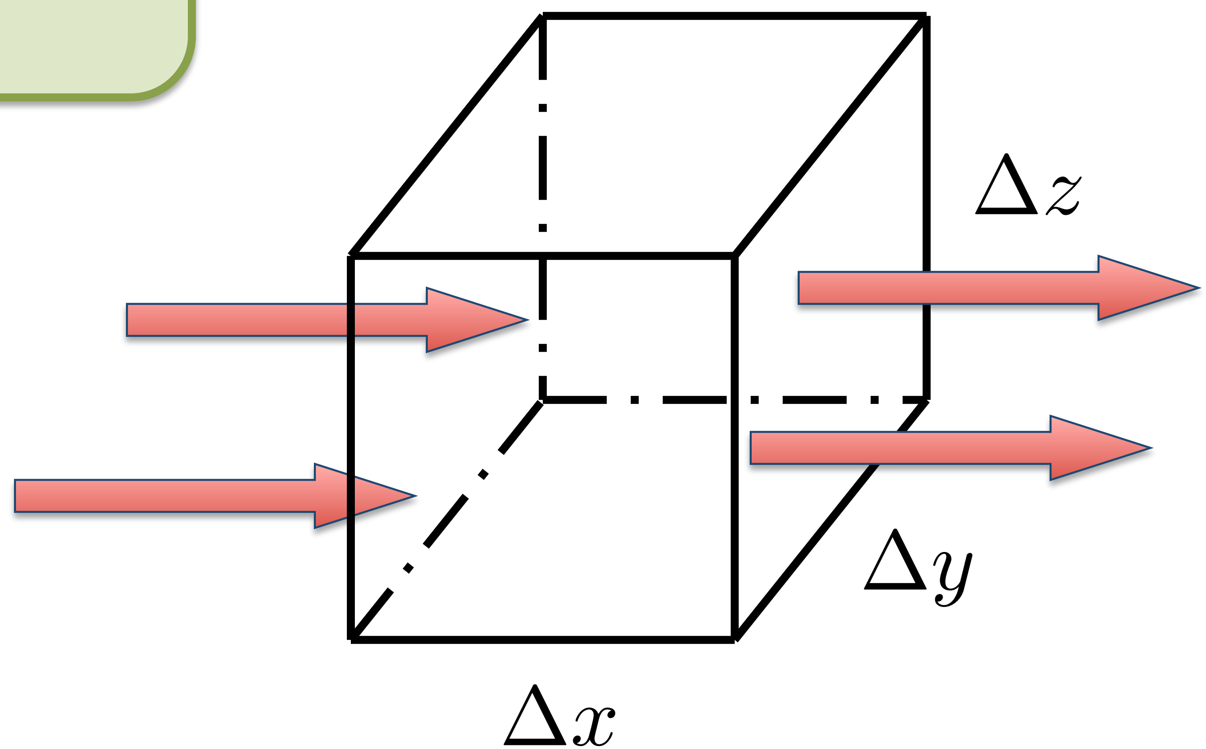
$$\boxed{\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{u})}$$

Recognize the **divergence**.

Continuity Equation: Eulerian

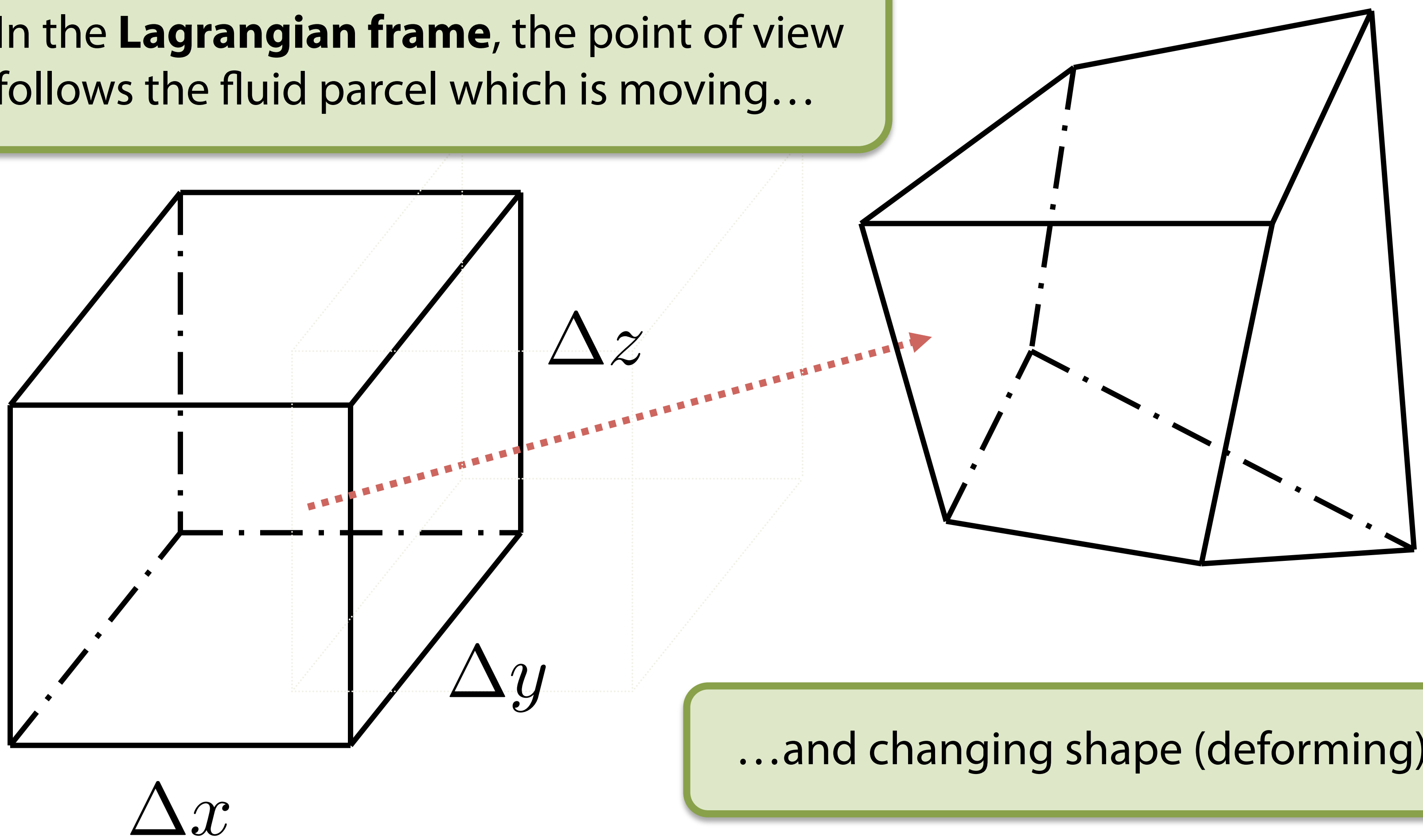
In the **Eulerian frame**, a region of the atmosphere is a **fixed volume** and fluid flows through it.

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{u})$$



Question: What is the **continuity equation** in the Lagrangian frame?

In the **Lagrangian frame**, the point of view follows the fluid parcel which is moving…



…and changing shape (deforming)

Continuity Equation: Lagrangian

Starting with the continuity equation in the Eulerian frame

$$\boxed{\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{u})}$$

Apply product rule (recall your vector identities)

$$\nabla \cdot (\rho \mathbf{u}) = \rho \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla \rho$$

Collect terms:

$$\boxed{\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho} = -\rho \nabla \cdot \mathbf{u}$$

What is this?

Continuity Equation: Lagrangian

$$\boxed{\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho} = -\rho \nabla \cdot \mathbf{u}$$

Material Derivative

$$\boxed{\frac{1}{\rho} \frac{D\rho}{Dt} = -\nabla \cdot \mathbf{u}}$$

Evolution equation for density in the **Lagrangian Frame**

Continuity Equation: Lagrangian

$$\frac{1}{\rho}\frac{D\rho}{Dt} = -\nabla \cdot \mathbf{u}$$

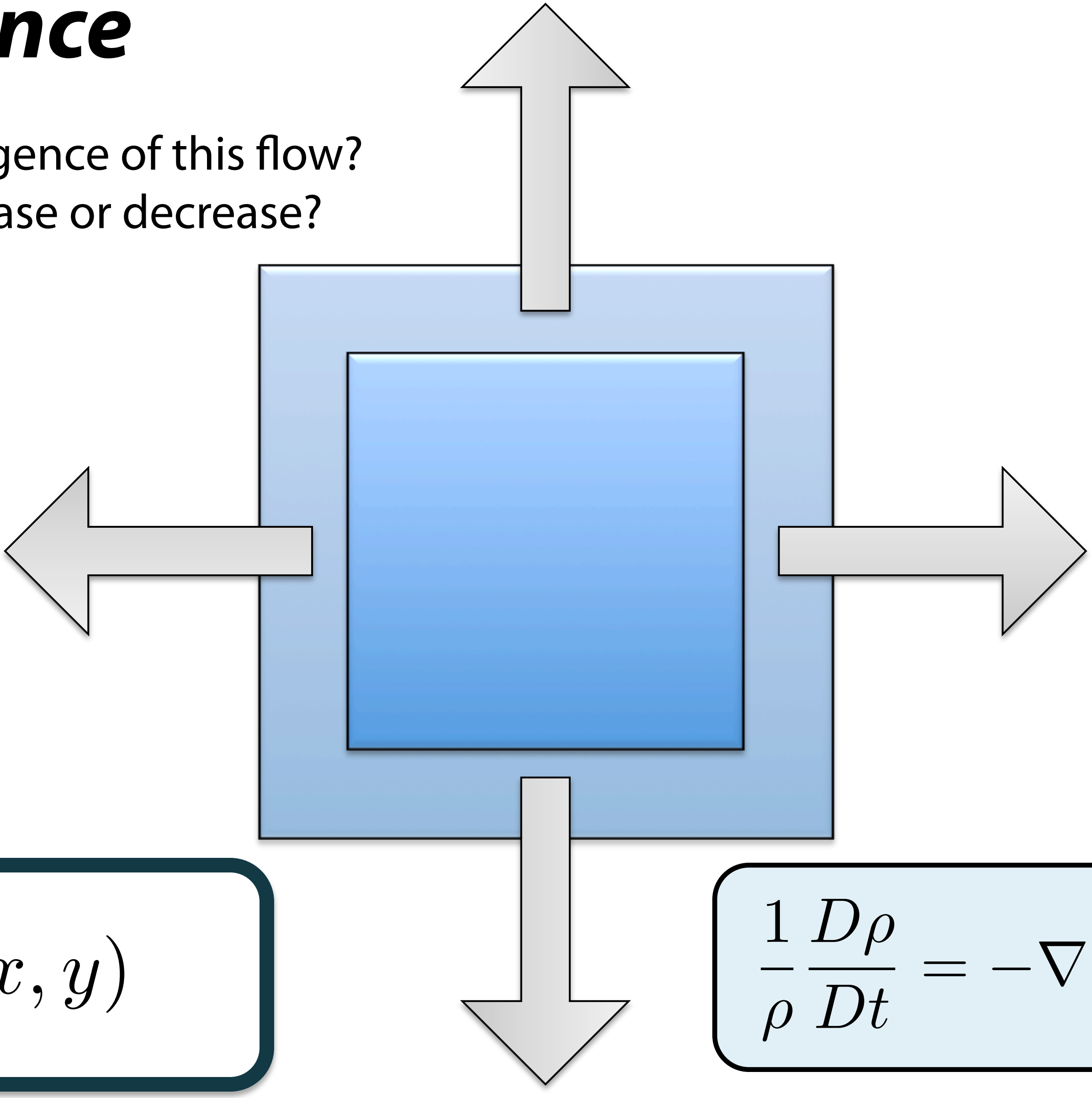
The change in density following the motion is proportional to the **divergence**

Convergence = increase in density (compression)

Divergence = decrease in density (expansion)

Divergence

What is the divergence of this flow?
Will density increase or decrease?
Calculate it!

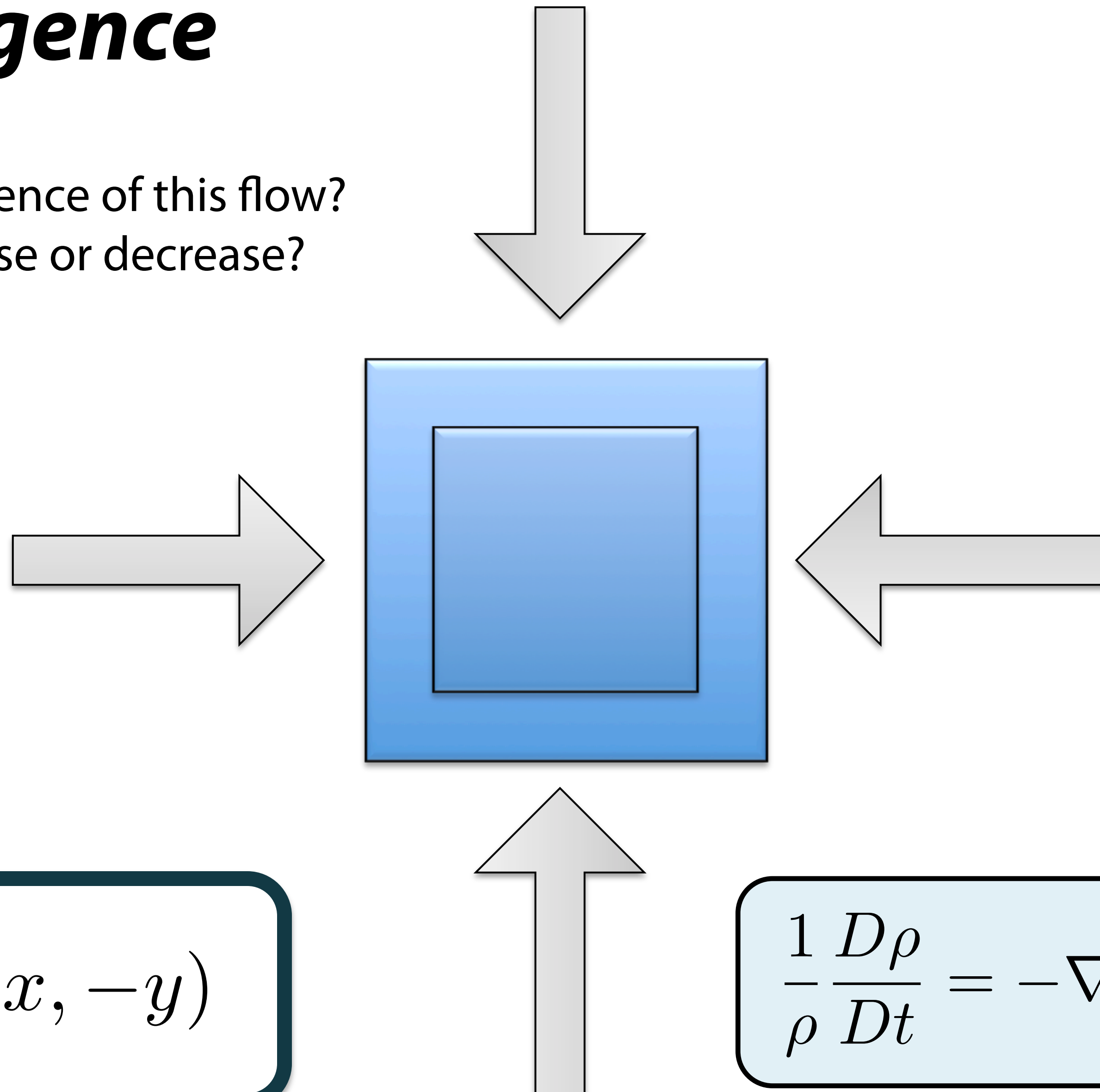


$$\mathbf{u} = (x, y)$$

$$\frac{1}{\rho}\frac{D\rho}{Dt} = -\nabla \cdot \mathbf{u}$$

Convergence

What is the divergence of this flow?
Will density increase or decrease?
Calculate it!

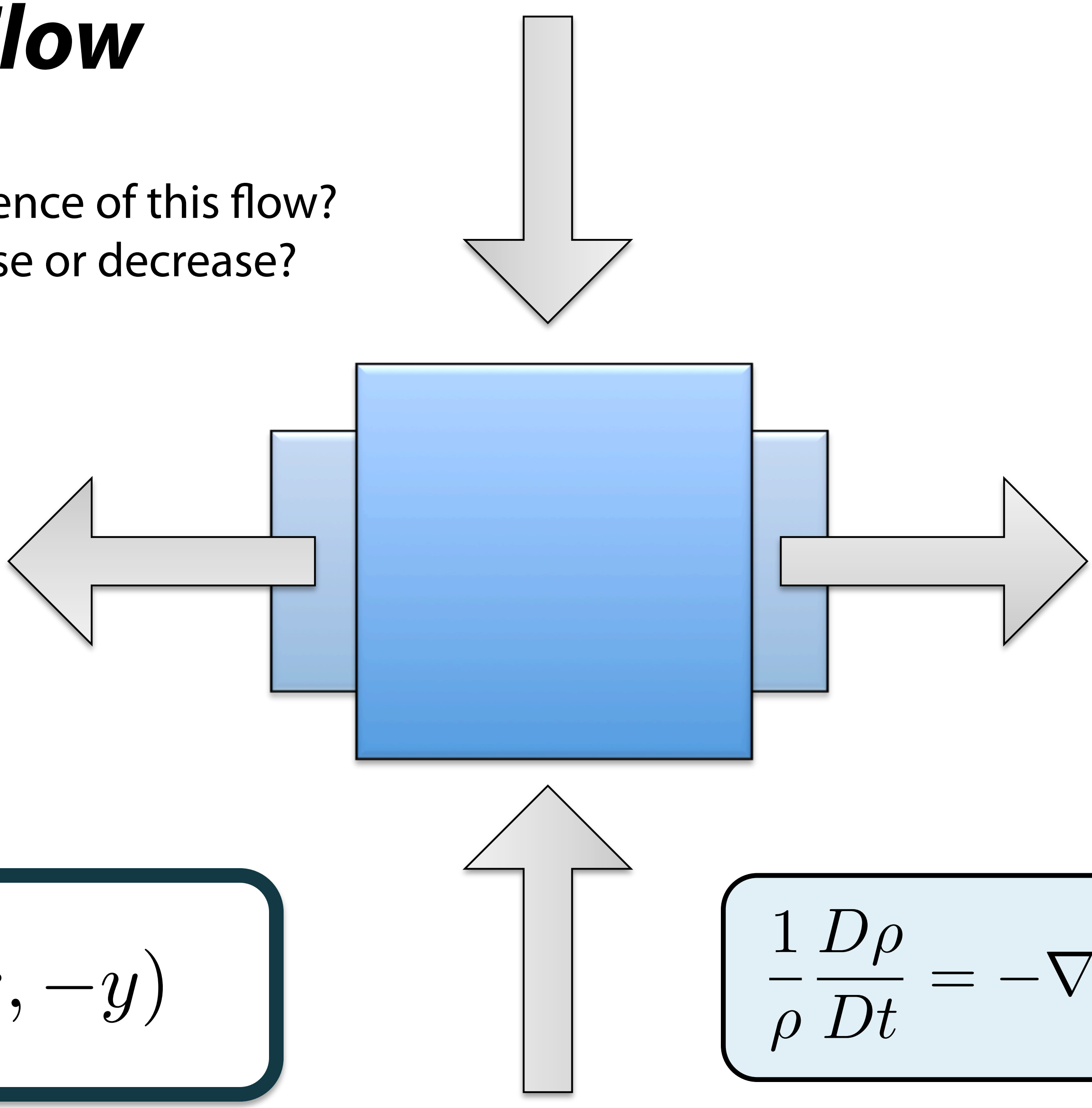


$$\mathbf{u} = (-x, -y)$$

$$\frac{1}{\rho}\frac{D\rho}{Dt} = -\nabla \cdot \mathbf{u}$$

Mixed Flow

What is the divergence of this flow?
Will density increase or decrease?
Calculate it!



$$\mathbf{u} = (x, -y)$$

$$\frac{1}{\rho}\frac{D\rho}{Dt} = -\nabla \cdot \mathbf{u}$$

Tracer Transport

Passive tracers (chemical species, moisture) are transported with the flow. Their mixing ratio (the ratio of tracer mass to fluid mass) is constant within a fluid parcel:

$$\boxed{\frac{Dq_i}{Dt} = 0 \qquad \text{(Advection equation)}}$$

To write a conservation law for tracers, we use the continuity equation:

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = -\rho \nabla \cdot \mathbf{u}$$

$$\frac{\partial \rho}{\partial t} q_i + q_i \mathbf{u} \cdot \nabla \rho = -\rho q_i \nabla \cdot \mathbf{u}$$

$$\rho \frac{\partial q_i}{\partial t} + \rho \mathbf{u} \cdot \nabla q_i = 0$$

$$\boxed{\frac{\partial}{\partial t}(\rho q_i) + \nabla \cdot (\rho q_i \mathbf{u}) = 0 \qquad \text{(Flux-Form)}}$$

Thanks!