# Introduction to Atmospheric Dynamics

## Chapter 1

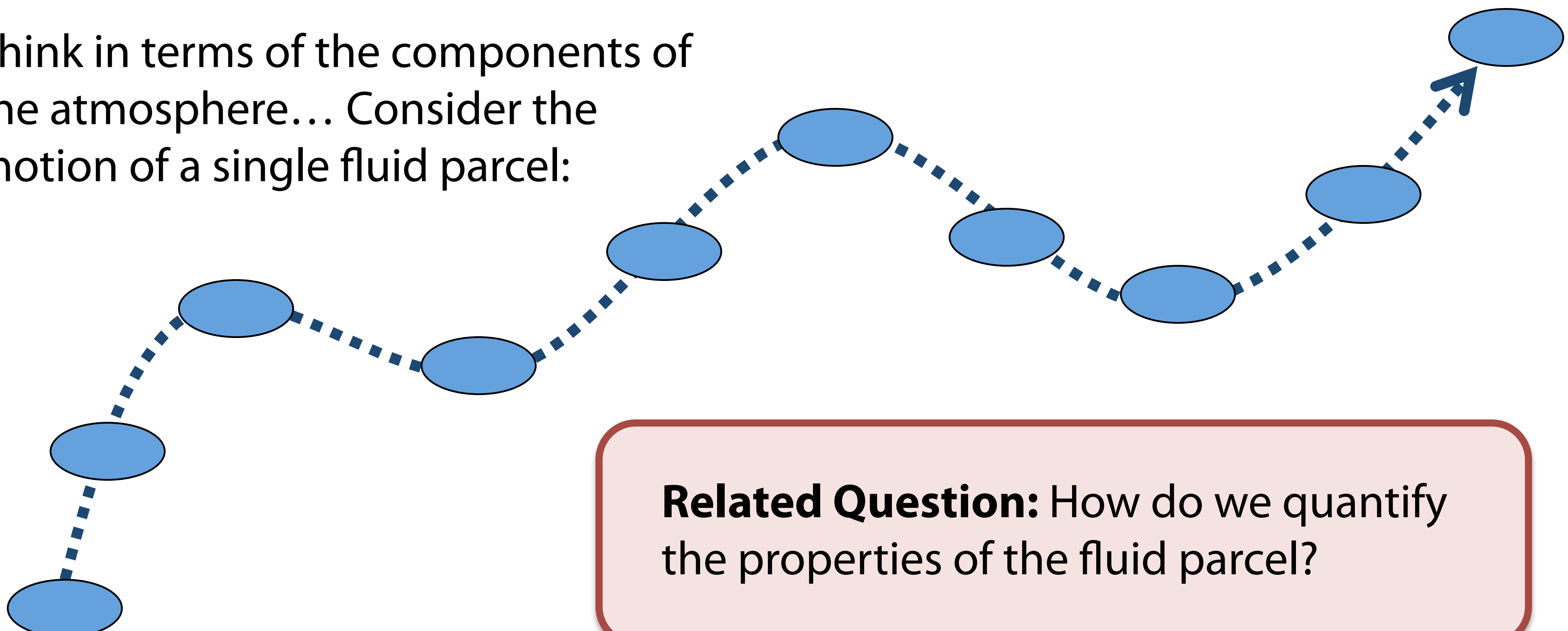
**Paul A. Ullrich**
paullrich@ucdavis.edu

# Part 2: The Material Derivative

# *Reference Frames*

**Question:** How do we quantify the properties of the global atmosphere?

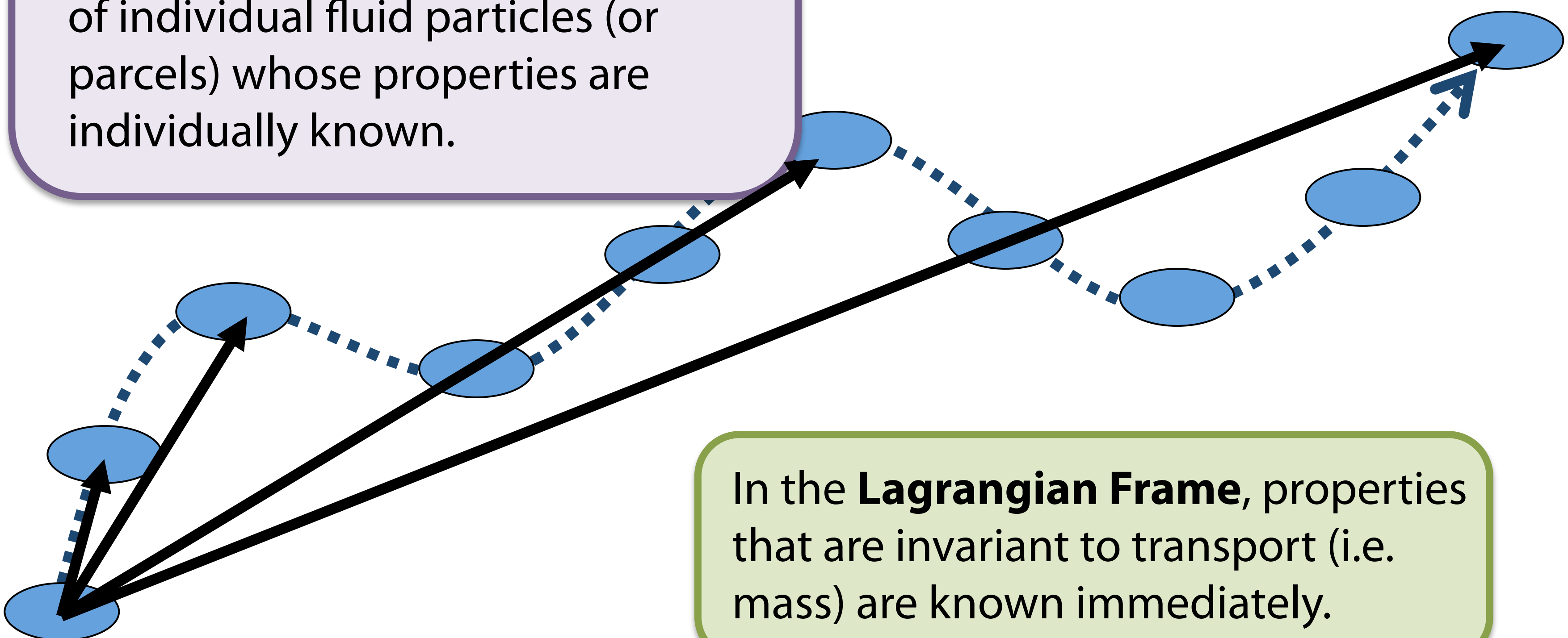
Think in terms of the components of the atmosphere… Consider the motion of a single fluid parcel:



**Related Question:** How do we quantify the properties of the fluid parcel?

# The Lagrangian Frame

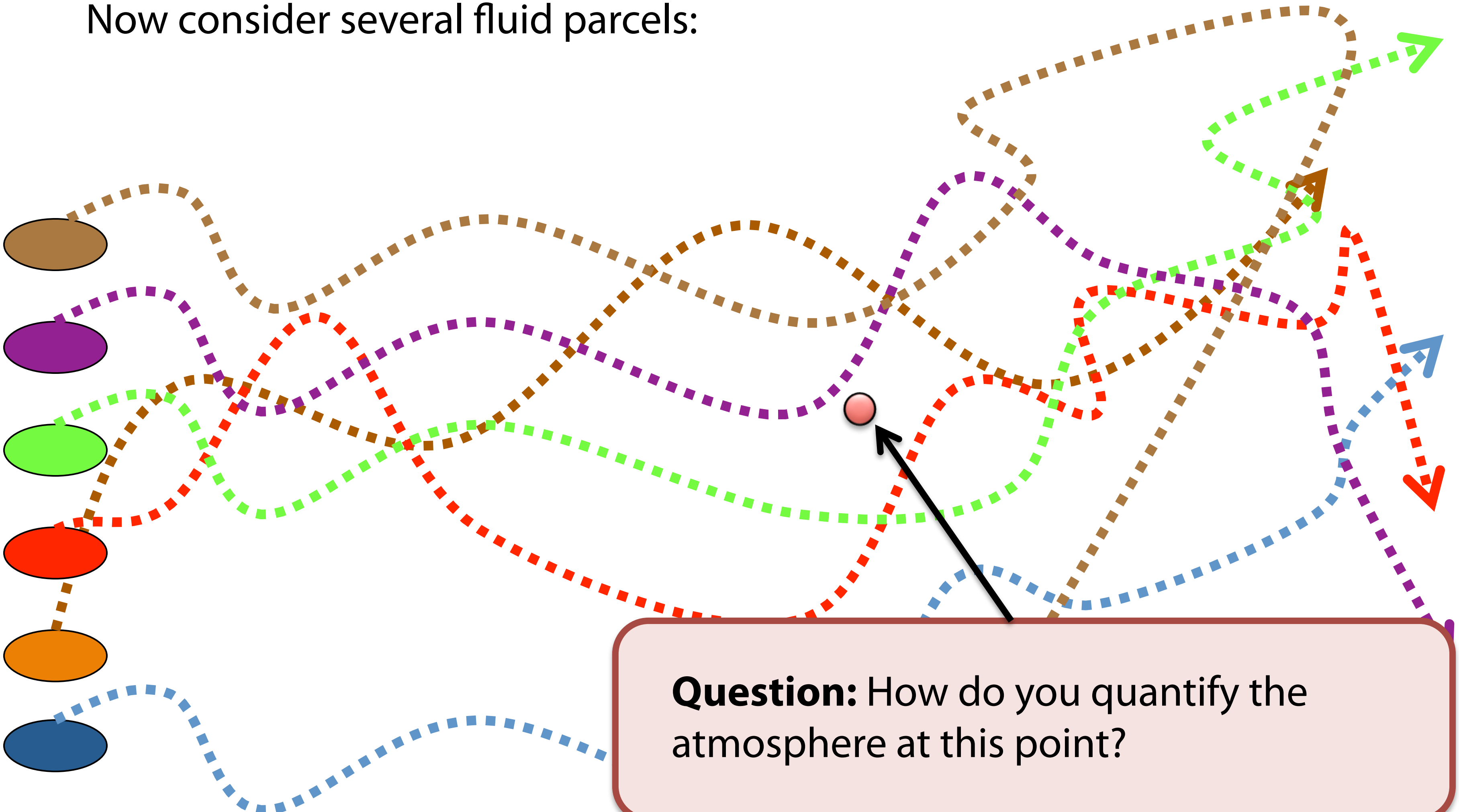
**Definition:** In the **Lagrangian Frame** the properties of the whole atmosphere are described in terms of individual fluid particles (or parcels) whose properties are individually known.



In the **Lagrangian Frame**, properties that are invariant to transport (i.e. mass) are known immediately.

# The Lagrangian Frame

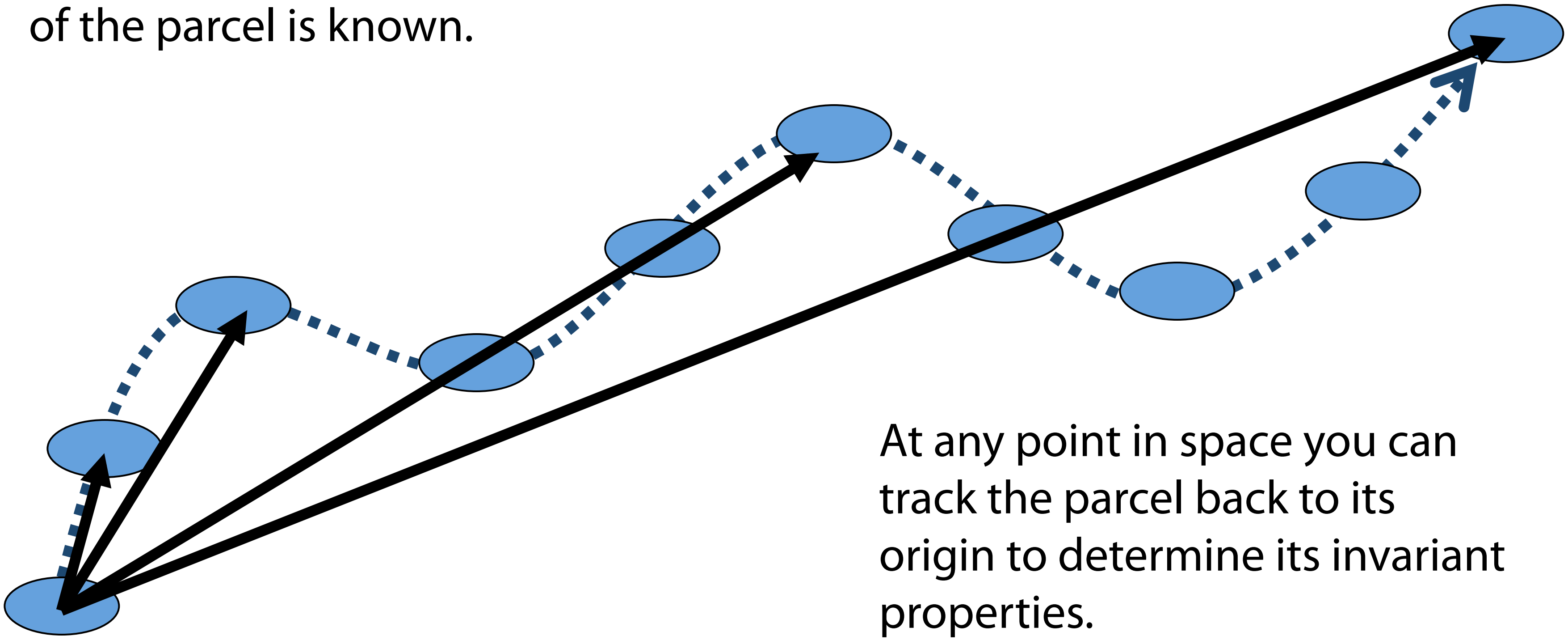
Now consider several fluid parcels:



**Question:** How do you quantify the atmosphere at this point?

# The Lagrangian Frame

**Idea:** Use a position vector that changes in time. The parcel position is a function of its starting point, so the history of the parcel is known.



At any point in space you can track the parcel back to its origin to determine its invariant properties.

# *The Lagrangian Frame*

Under the ***Lagrangian*** point of view:

- Benefits:
  - Useful for developing theory
  - Very powerful for visualizing fluid motion
  - The history of each fluid parcel is known

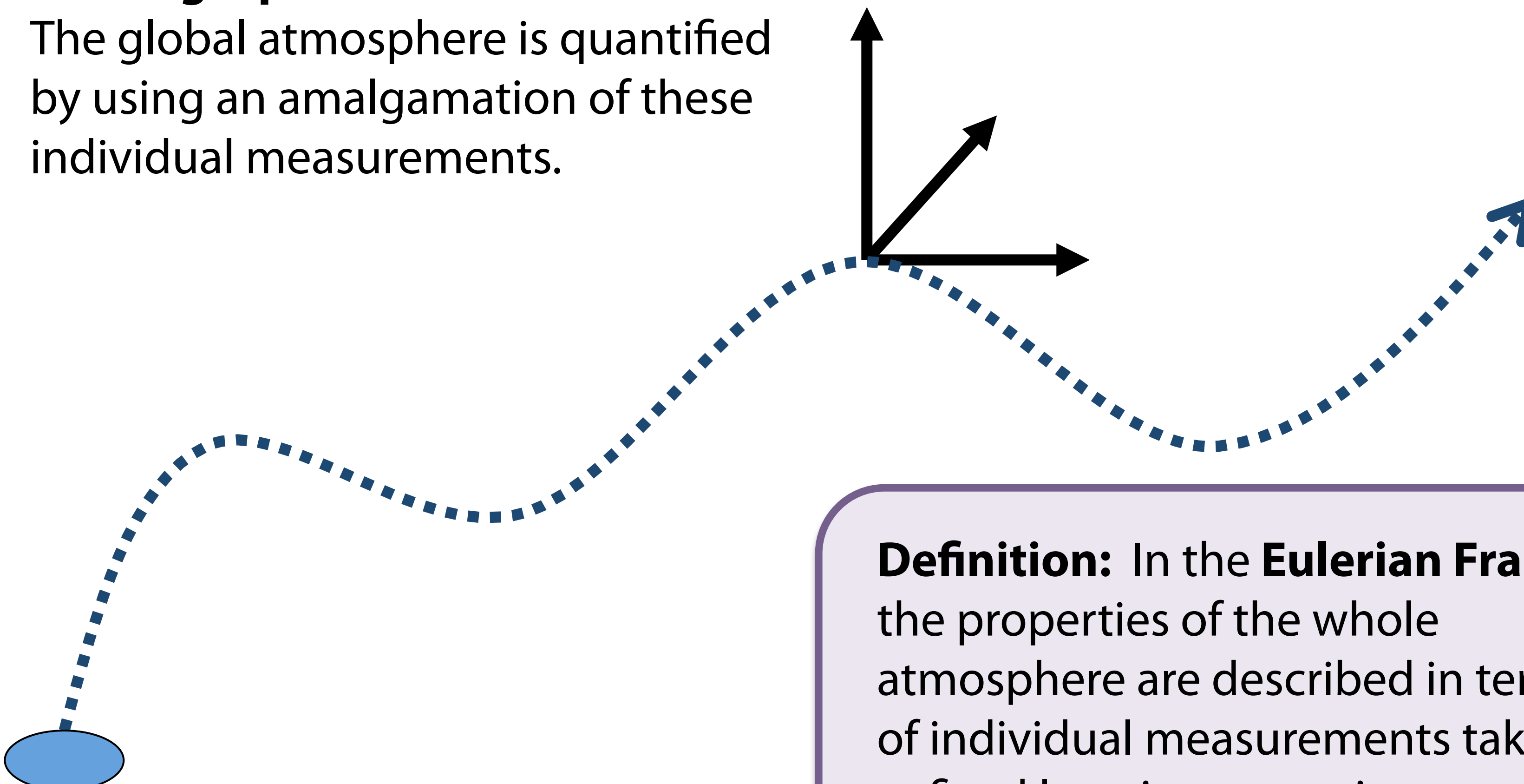
- Problems:
  - Need to keep track of a coordinate system for each parcel
  - How do you account for interactions between different parcels?
  - What can we say about regions where fluid parcels are sparse?
  - What can we say about the fluid as a whole if all of the initial parcels bunch together?

# The Lagrangian Frame

**Weather balloons** are a source of measurements in the Lagrangian frame. They are passively transported by the background wind field (and so are essentially isolated fluid parcels).

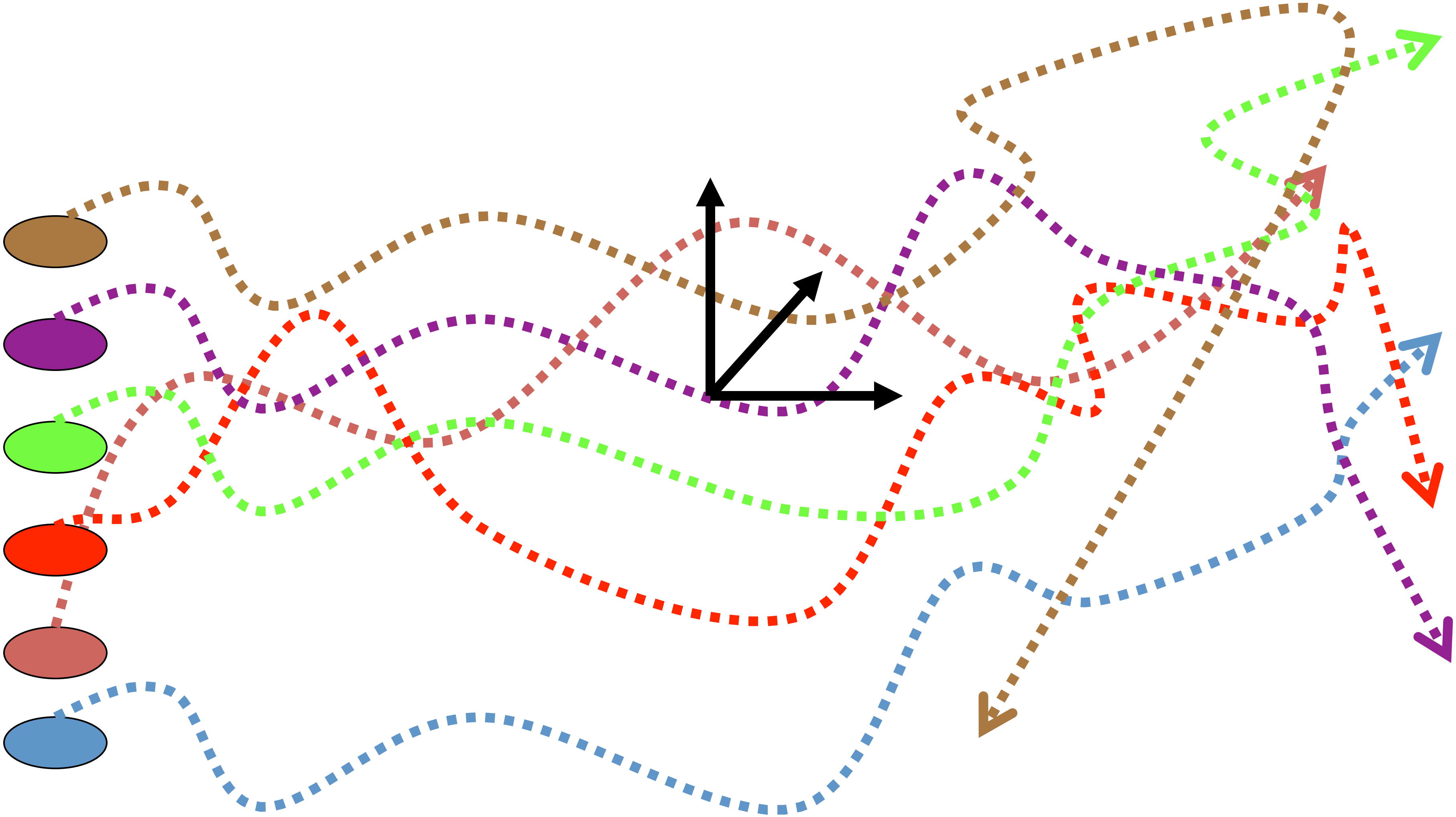
# *The Eulerian Frame*

**Idea:** Measurements could be taken at a **single point** as a function of time. The global atmosphere is quantified by using an amalgamation of these individual measurements.



**Definition:** In the **Eulerian Frame** the properties of the whole atmosphere are described in terms of individual measurements taken at fixed locations over time.

# How would we quantify this?



# The Eulerian Frame

Under the ***Eulerian*** point of view.

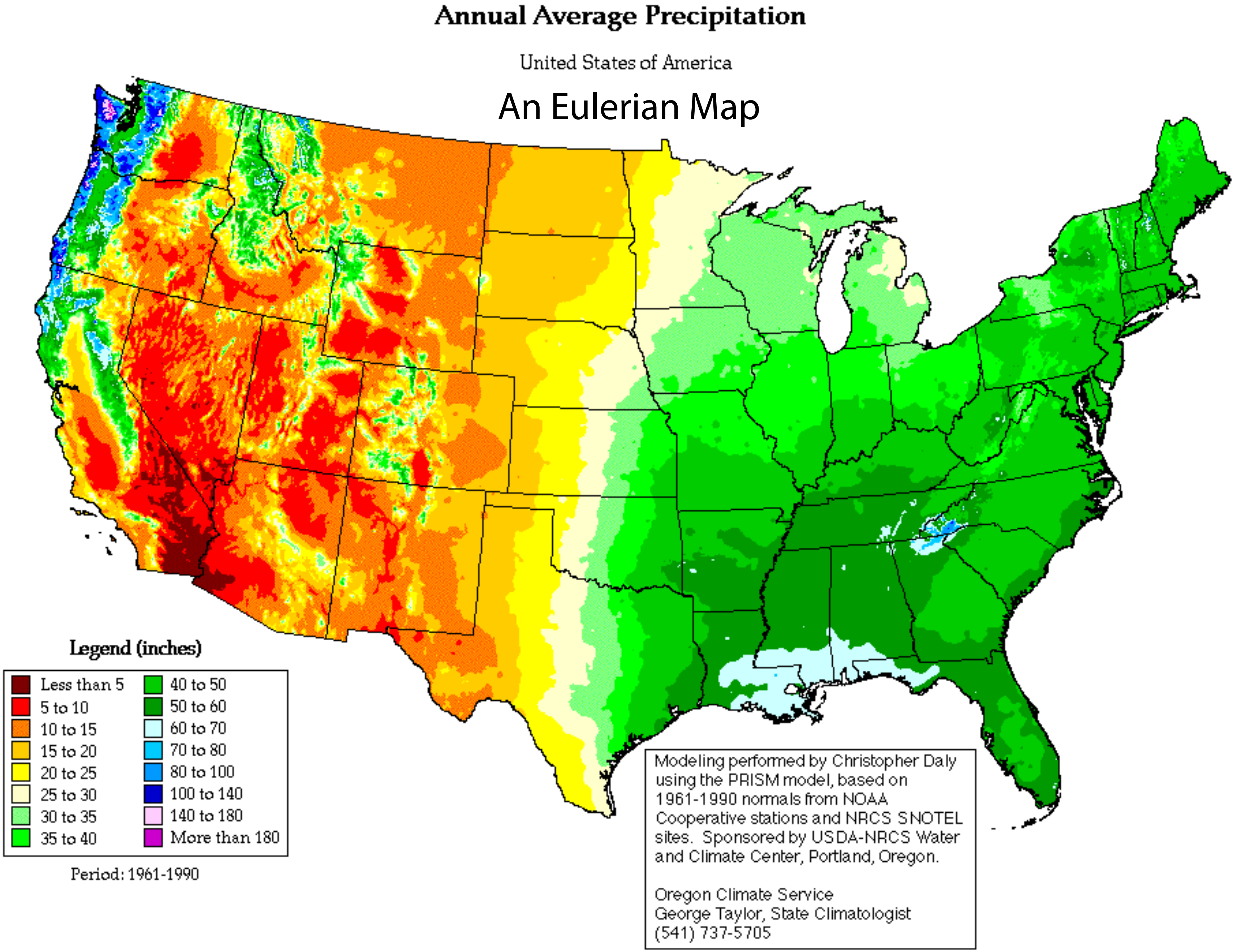
- Benefits:
  - Useful for developing theory
  - Requires considering only one coordinate system for all parcels
  - Easy to represent interactions of parcels through surface forces
  - Looks at the fluid as a field.
  - A value for each point in the field – no gaps or bundles of "information."

- Problems
  - More difficult to keep track of parcel history—not as useful for applications such as pollutant dispersion (or clouds?)…

# The Eulerian Frame

**Weather stations** are a source of measurements in the Eulerian frame.  They are placed in a fixed location and observe the properties of fluid parcels as they pass by the instruments.

# An Eulerian Map



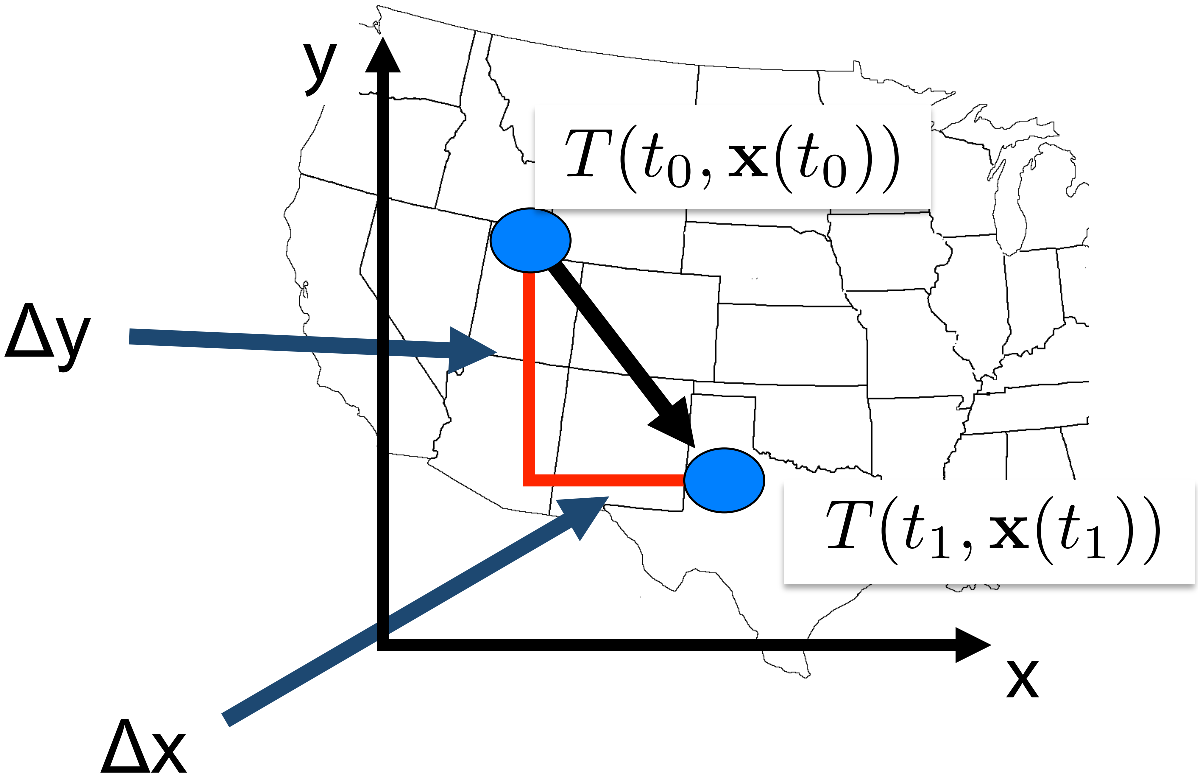
**Question:** Why consider two frames of reference?

**Answer:** Since the same physical principles hold regardless of the reference frame, the use of multiple reference frames is primarily for purposes of understanding.

Certain concepts can be more easily explained in the Lagrangian frame, whereas others are better explained in the Eulerian frame.

# The Material Derivative

Consider a parcel with some property of the atmosphere, like temperature ($T$), that moves some distance in time Δt.




$$\Delta T = T(t_1, \mathbf{x}(t_1)) - T(t_0, \mathbf{x}(t_0))$$

# *The Material Derivative*

We would like to calculate the change in temperature over time Δt, following the parcel.

Expand the change in temperature in a Taylor series around the temperature at the initial position.

$$\Delta T = \frac{\partial T}{\partial t}\Delta t + \frac{\partial T}{\partial x}\Delta x + \frac{\partial T}{\partial y}\Delta y + \frac{\partial T}{\partial z}\Delta z + \boxed{\text{Higher Order Terms}}$$

Assume increments over $\Delta t$ are small, and ignore Higher Order Terms

# The Material Derivative

$$\Delta T = \frac{\partial T}{\partial t}\Delta t + \frac{\partial T}{\partial x}\Delta x + \frac{\partial T}{\partial y}\Delta y + \frac{\partial T}{\partial z}\Delta z$$

Divide through by $\Delta t$

$$\frac{\Delta T}{\Delta t} = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial T}{\partial y}\frac{\Delta y}{\Delta t} + \frac{\partial T}{\partial z}\frac{\Delta z}{\Delta t}$$

Take the limit for small $\Delta t$

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}\frac{Dx}{Dt} + \frac{\partial T}{\partial y}\frac{Dy}{Dt} + \frac{\partial T}{\partial z}\frac{Dz}{Dt}$$

# The Material Derivative

Remember, by definition:

$$\frac{Dx}{Dt} = u, \quad \frac{Dy}{Dt} = v, \quad \frac{Dz}{Dt} = w$$

… so the material derivative becomes

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}$$

Lagrangian Frame

Eulerian Frame

Change due to **advection**

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T$$

Coordinate independent form

# Material Derivative

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}$$

Lagrangian Frame

Eulerian Frame

Change due to **advection**

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T$$

Coordinate independent form

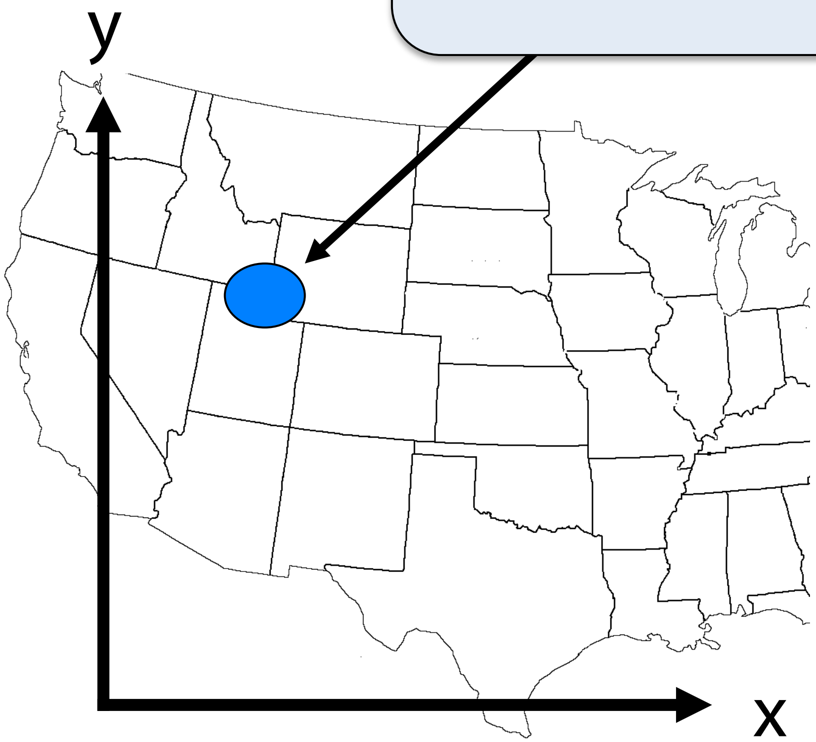
This formula connects the **Lagrangian frame** (which describes the properties of fluid parcels) and the **Eulerian frame** (which describes the properties at particular locations).

**Question:** What is the change in temperature at a point?

$$\frac{\partial T}{\partial t} = \frac{DT}{Dt} - \mathbf{u} \cdot \nabla T$$

Eulerian Frame

$T$ change at a fixed point (x, y, z held constant)

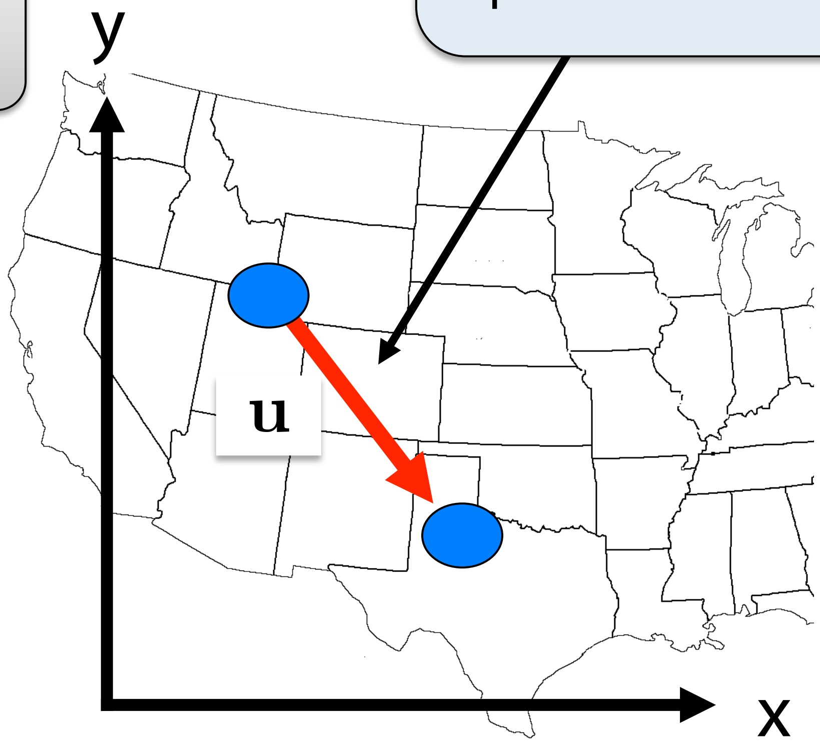


**Question:** What is the change in temperature at a point?

$$\frac{\partial T}{\partial t} = \frac{DT}{Dt} - \mathbf{u} \cdot \nabla T$$

Lagrangian Frame

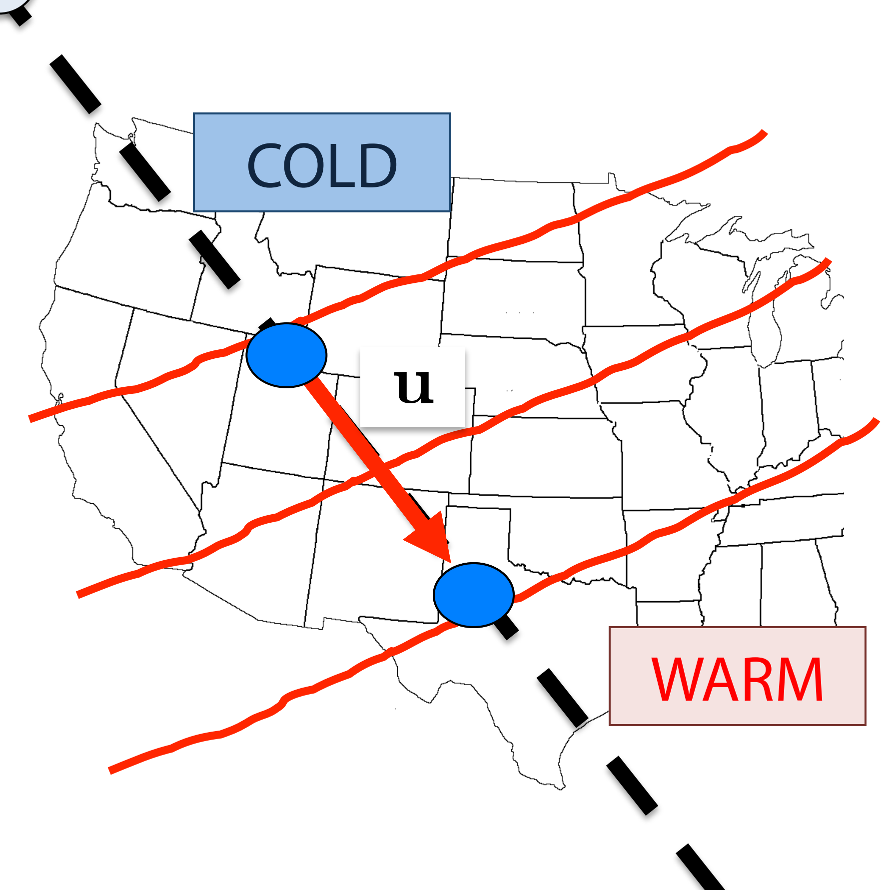
Material derivative, $T$ change following the parcel



**Question:** What is the change in temperature at a point?

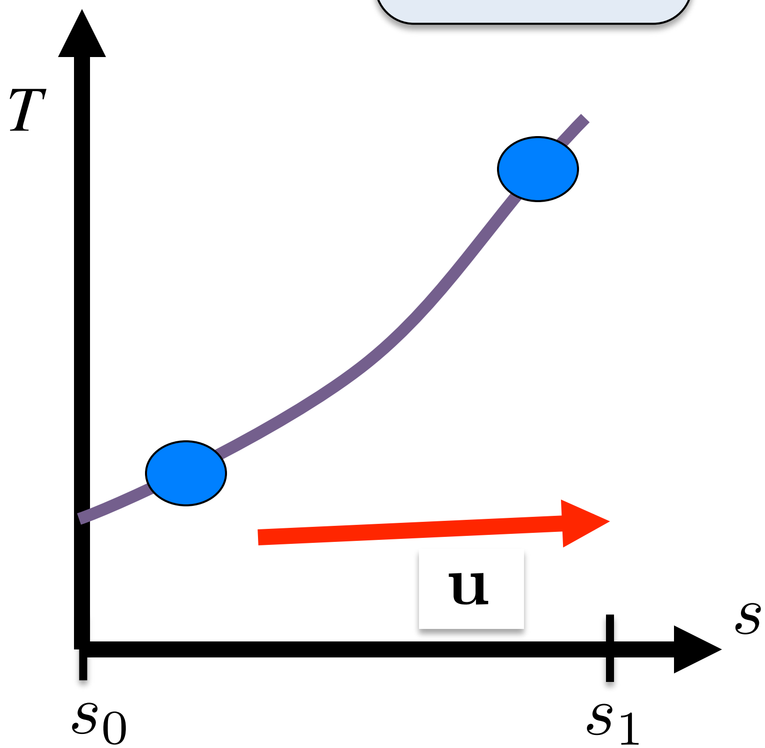
$$\frac{\partial T}{\partial t} = \frac{DT}{Dt} - \mathbf{u} \cdot \nabla T$$

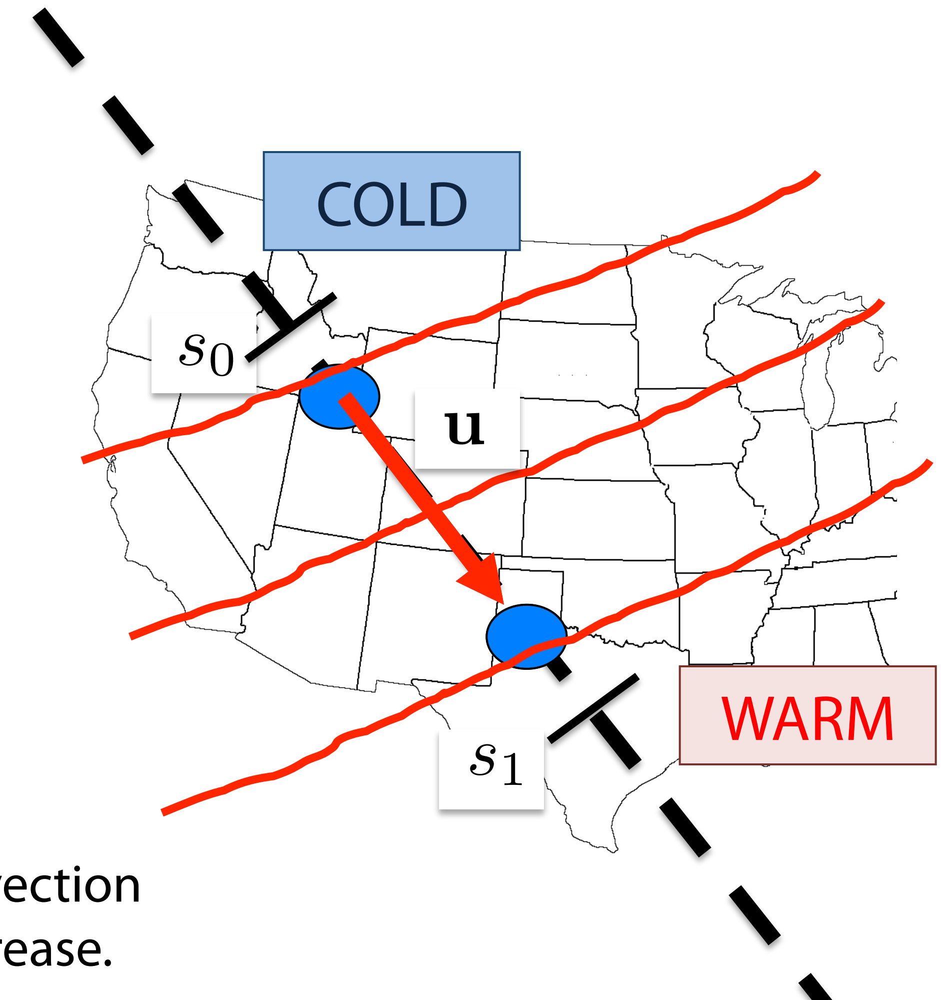
**Advection:** This is the same as taking the derivative of $T$ along the line defined by the velocity vector and multiplying by (-1).



**Question:** What is the change in temperature at a point?

$$\frac{\partial T}{\partial t} = \frac{DT}{Dt} - \mathbf{u} \cdot \nabla T$$



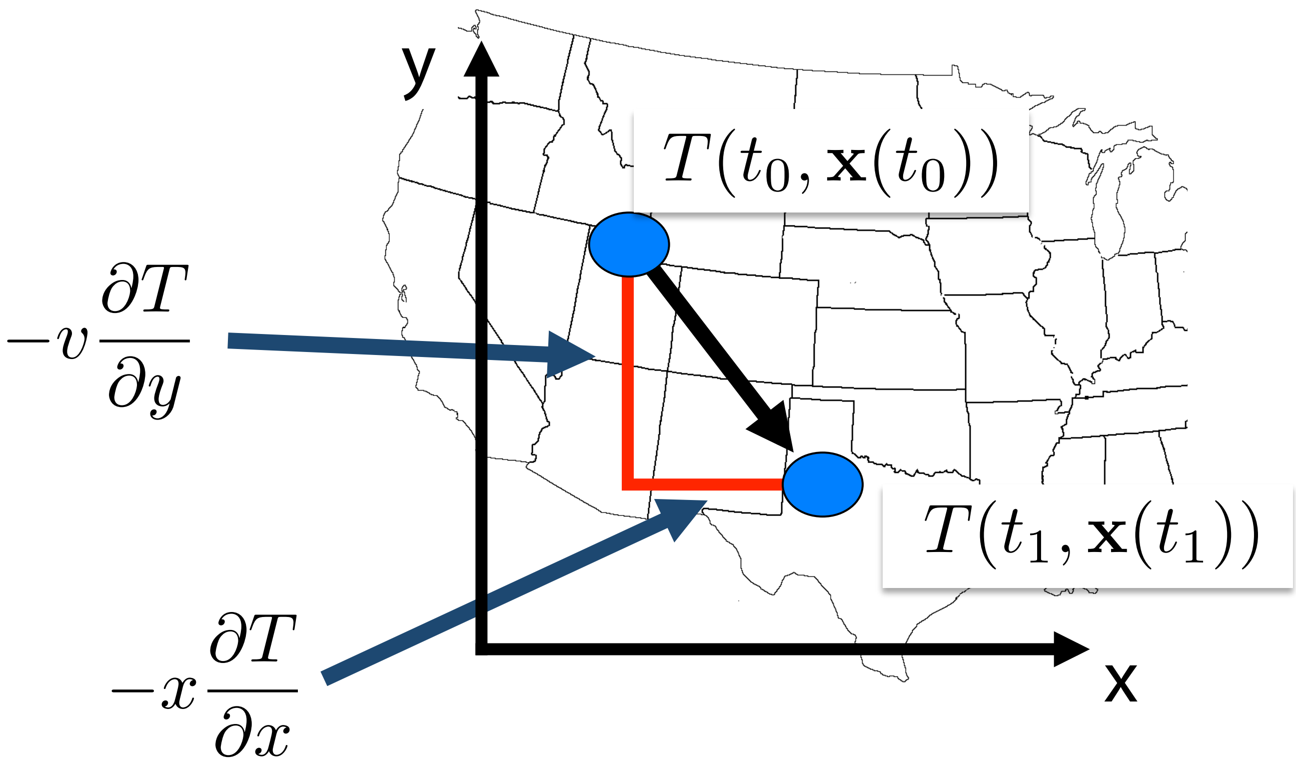




If dT/ds is positive then advection causes temperature to decrease.

**Question:** What is the change in temperature at a point?

Expanding advection into its components, we have

$$-\mathbf{u} \cdot \nabla T = -u\frac{\partial T}{\partial x} - v\frac{\partial T}{\partial y} - w\frac{\partial T}{\partial z}$$



# The Momentum Equation

Remember, we derived from force balances

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u} - g\mathbf{k} + \mathrm{Co} + \mathrm{Curv}$$

This is the momentum equation in the Lagrangian frame

In the Eulerian reference frame, we have via Material Derivative

$$\frac{\partial\mathbf{u}}{\partial t} = \boxed{-\mathbf{u}\cdot\nabla\mathbf{u}} - \frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u} - g\mathbf{k} + \mathrm{Co} + \mathrm{Curv}$$

Non-linear advection of wind velocity