Introduction to Atmospheric Dynamics

Chapter 1

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How to Read These Slides

Definition: A definition is an explanation or outline for relevant jargon or terms.

Concept: An idea that draws a connection between subjects or provides an answer for a question.

Question: What is something that motivates delving into this topic?

Part 1: Forces in the Atmosphere

The Earth's Atmosphere

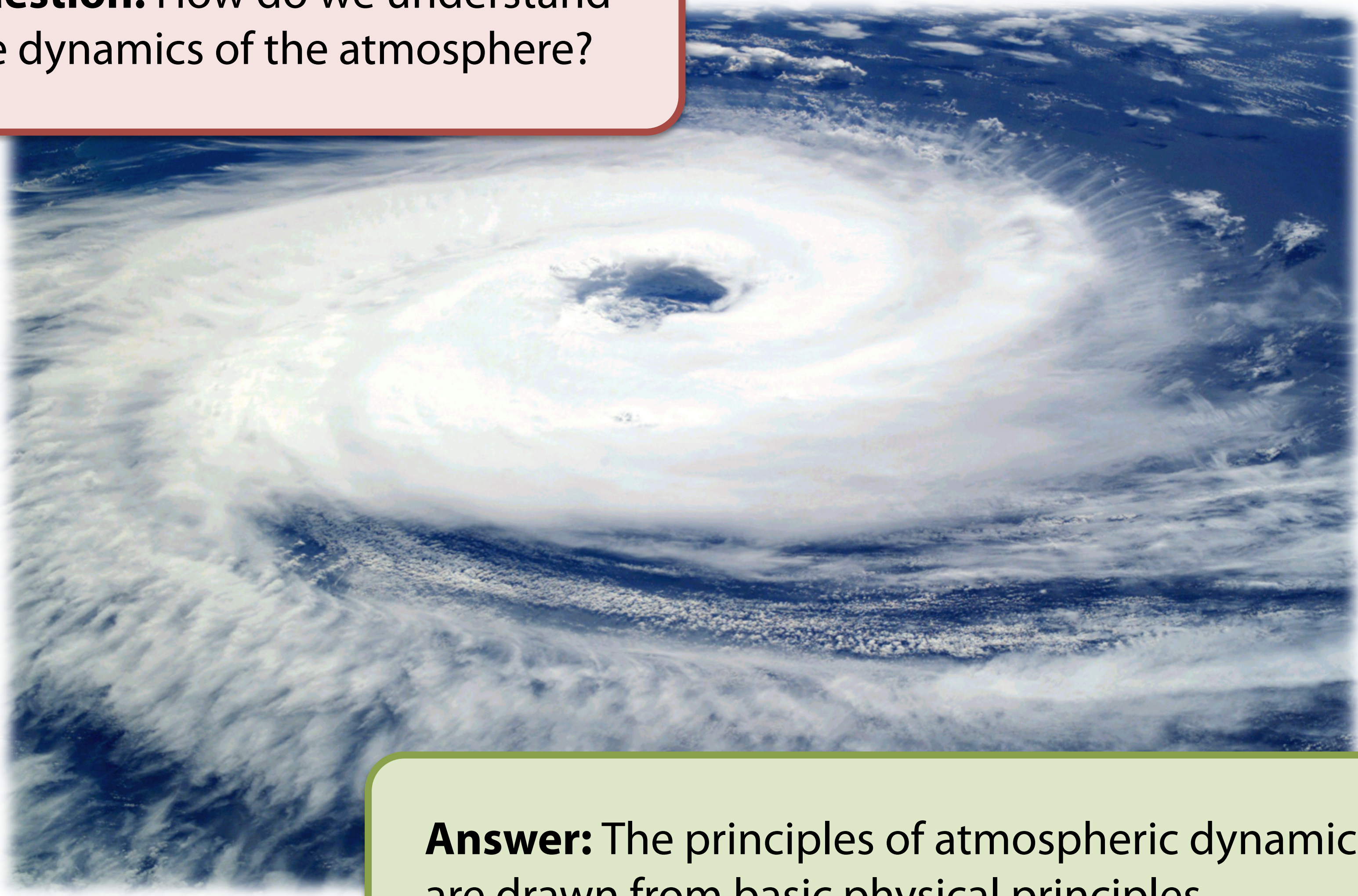
Radius of the Earth
6371.22 km

Atmosphere Depth
100 km

Troposphere Depth
10 km

Mountain Height
8.8 km

Question: How do we understand the dynamics of the atmosphere?



Answer: The principles of atmospheric dynamics are drawn from basic physical principles.

Question: What are the basic physical principles that govern the atmosphere?

Newton's Second Law: The change in momentum of an object is equal to the sum of forces acting on that object.

$$\frac{d(m\mathbf{v})}{dt} = \sum_{\text{all } i} \mathbf{F}_i$$

Conservation of Momentum: With no external forces momentum must be conserved.

Basic Principles of Physics

Definition: Velocity is the change of position with respect to time

$$\mathbf{u} = \frac{d\mathbf{x}}{dt}$$

Definition: Acceleration is the change of velocity with respect to time

$$\mathbf{a} = \frac{d\mathbf{u}}{dt}$$

Hence, for an **object of constant mass**:

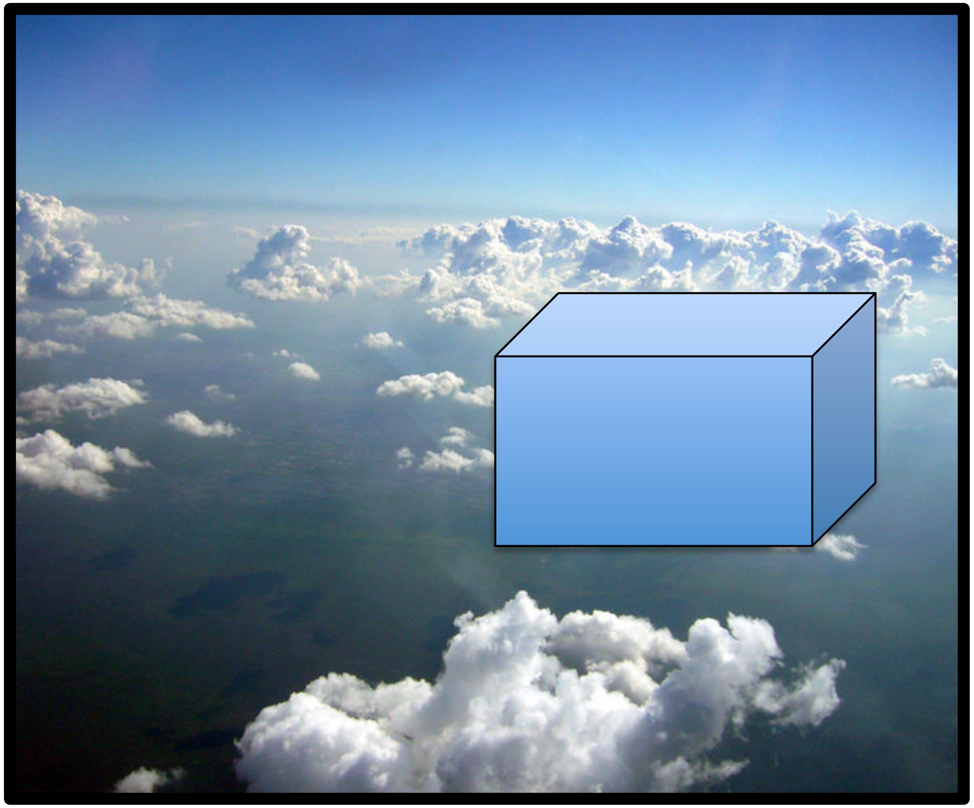
$$\frac{d(m\mathbf{u})}{dt} = m\frac{d\mathbf{u}}{dt} = m\mathbf{a}$$

Newton's Second Law

$$\boxed{\mathbf{a} = \frac{1}{m}\sum_{\text{all } i} \mathbf{F}_i}$$

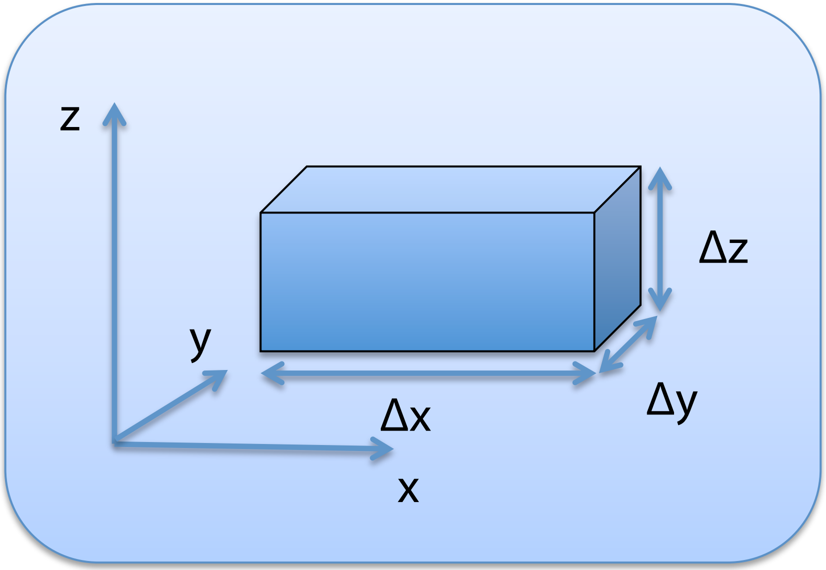
Basic Principles of Physics

- How do these forces induce acceleration?
- We assume the existence of an idealized "parcel" of fluid.
- Forces are calculated on the idealized parcel.
- Then take the limit of the parcel being infinitely small.

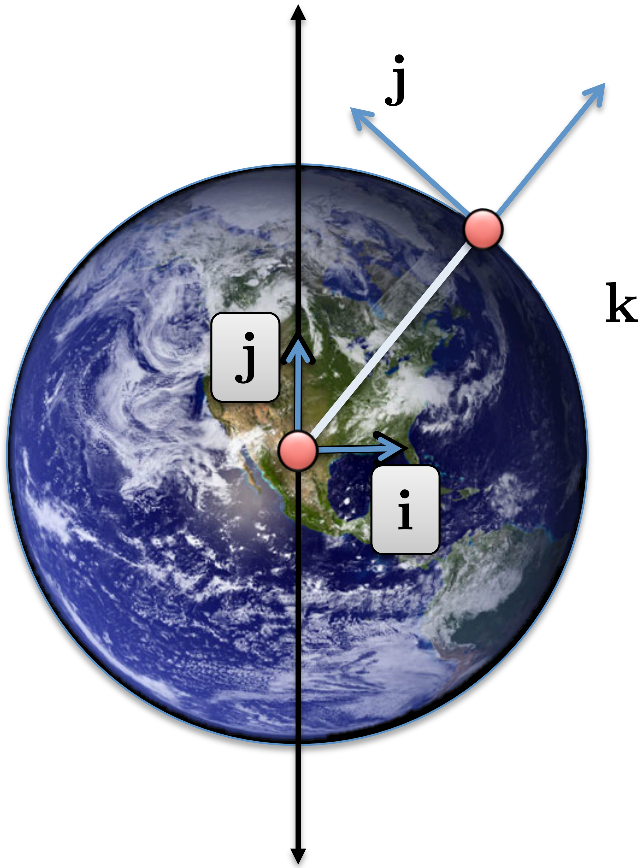


Parcel Properties

ρ	Density (kg/m^3)
$V = \Delta x \Delta y \Delta z$	Volume (m^3)
$m = \rho \Delta x \Delta y \Delta z$	Mass (kg)
T	Temperature (K)
p	Pressure (Pa)



Spherical Coordinates



Spherical coordinates:

λ	Longitude
ϕ	Latitude
r	Radius

Basis vectors:

$\mathbf{i}$	Eastward basis vector
$\mathbf{j}$	Northward basis vector
$\mathbf{k}$	Vertical basis vector

Definition: The **Material Derivative** (expressed with a capital D) denotes the change in a quantity **following a fluid parcel**.

$$\mathbf{a} = \frac{1}{m} \sum_{\text{all } i} \mathbf{F}_i \quad \Longleftrightarrow \quad \frac{D\mathbf{u}}{Dt} = \frac{1}{m} \sum_{\text{all } i} \mathbf{F}_i$$

$\mathbf{u} = (u, v, w)$	3D velocity vector
u	Eastward velocity (zonal velocity)
v	Northward velocity (meridional velocity)
w	Upward velocity (vertical velocity)

Question: What forces are important for understanding atmospheric dynamics?

$$\mathbf{a} = \frac{1}{m} \sum_{\text{all } i} \mathbf{F}_i \Longleftrightarrow \frac{D\mathbf{u}}{Dt} = \frac{1}{m} \sum_{\text{all } i} \mathbf{F}_i$$

- Pressure gradient force
- Gravitational force
- Viscous force
- Coriolis and centrifugal force

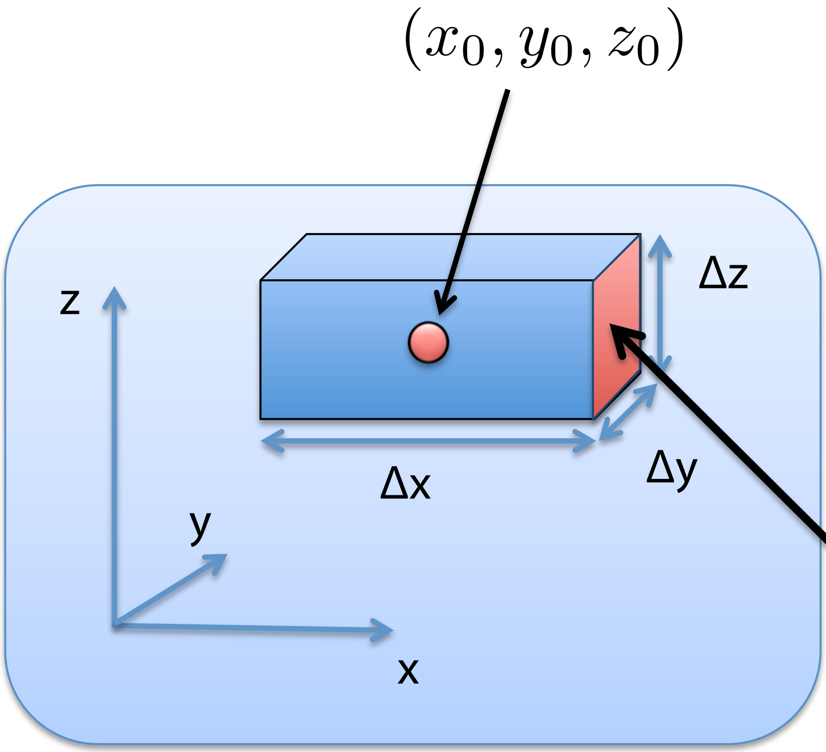
- Total force is the sum of all these forces

Definition: A **Surface Force** acts on the surface of a parcel of fluid, typically due to interactions with neighboring parcels. The magnitude of a surface force is typically proportional to the surface area of the parcel. **Examples:** Pressure Force, Viscous Force.

Definition: A **Body Force** acts on the center of mass of a parcel of fluid. The magnitude of the body force is typically proportional to the mass of the parcel. **Example:** Gravity.

Definition: When a coordinate system (for instance coordinates on the sphere) varies with respect to time and/or space, there is an **Apparent Force** due to the fact that coordinate vectors are changing following the fluid parcel. **Examples:** Coriolis, Centrifugal Force.

Pressure Gradient Force



Pressure at parcel center:

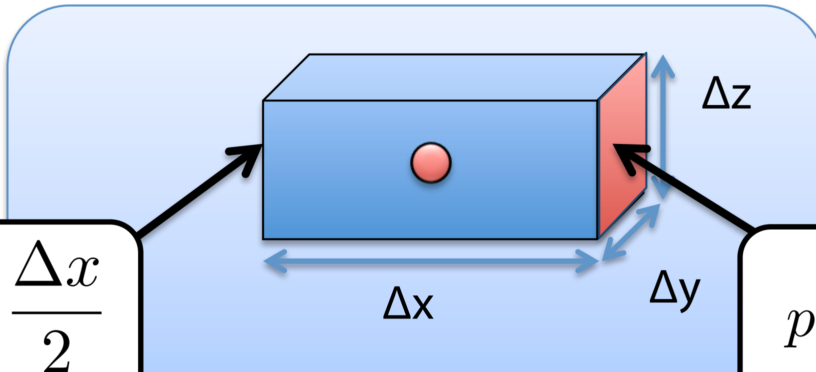
$$p_0 = p(x_0, y_0, z_0)$$

Approximate pressure here via Taylor expansion

$$p = p_0 + \left(\frac{\partial p}{\partial x}\right)\frac{\Delta x}{2} + \mathcal{O}(\Delta x^2)$$

Pressure Gradient Force

$$p_L \approx p_0 - \left(\frac{\partial p}{\partial x} \right) \frac{\Delta x}{2}$$



$$p_R \approx p_0 + \left(\frac{\partial p}{\partial x} \right) \frac{\Delta x}{2}$$

$$F_L = p_L A_L = p_L \Delta y \Delta z \qquad F_R = -p_R A_R = -p_R \Delta y \Delta z$$

$$\begin{aligned} F_{tot} &= F_L + F_R \\ &= \left[p_0 - \left(\frac{\partial p}{\partial x} \right) \frac{\Delta x}{2} \right] \Delta y \Delta z - \left[p_0 + \left(\frac{\partial p}{\partial x} \right) \frac{\Delta x}{2} \right] \Delta y \Delta z \\ &= - \left(\frac{\partial p}{\partial x} \right) \Delta x \Delta y \Delta z \end{aligned}$$

Total force acting on fluid parcel in the x direction

Pressure Gradient Force

Repeat in all coordinate directions:

$$\mathbf{F}_{tot} = -\left(\frac{\partial p}{\partial x}\mathbf{i} + \frac{\partial p}{\partial y}\mathbf{j} + \frac{\partial p}{\partial z}\mathbf{k}\right)\Delta x \Delta y \Delta z = -(\nabla p)\Delta x \Delta y \Delta z$$

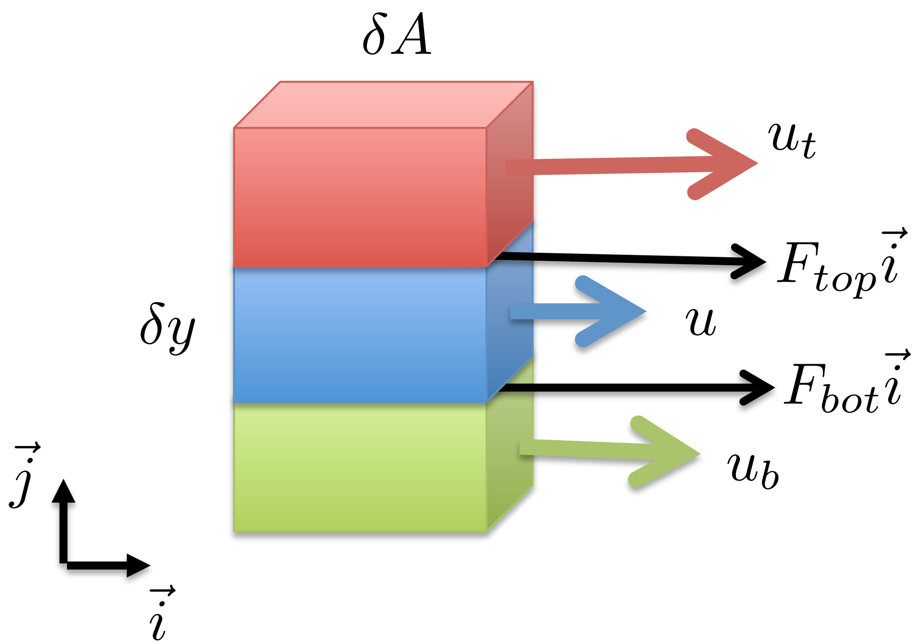
Then computing the force per unit mass (recall this determines the acceleration):

$$\boxed{\frac{\mathbf{F}_{tot}}{m} = -\frac{1}{\rho}\nabla p}$$

Total force acting on fluid parcel in all directions

Viscous Force

The ***viscosity*** of air is responsible for resisting motion of the fluid. It is a dissipative force, which results slowing a fluid which is not otherwise forced.



For example, the blue fluid parcel experiences a shear stress in the x direction due to motion of the red fluid parcel and green fluid parcel.

$$F_{top} \sim (u_t - u)\delta A$$

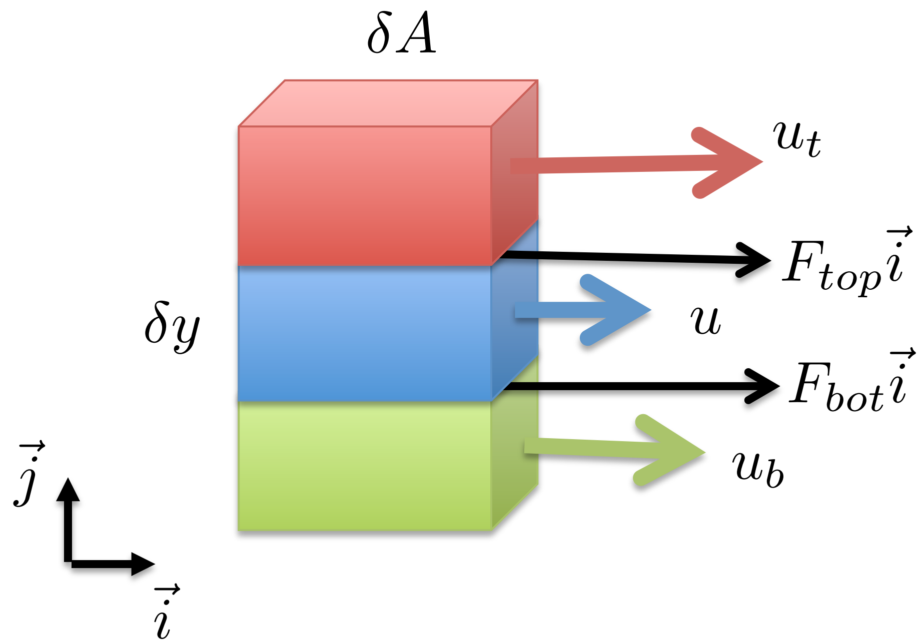
$$F_{bot} \sim (u_b - u)\delta A$$

Observe that if all fluid parcels are traveling at the same velocity, no force will be conferred.

Viscous Force

$$F_{top} \sim (u_t - u)\delta A$$

$$F_{bot} \sim (u_b - u)\delta A$$



The resistance of the fluid parcel to shearing is determined by the proportionality coefficient

$$\mu/\delta y$$

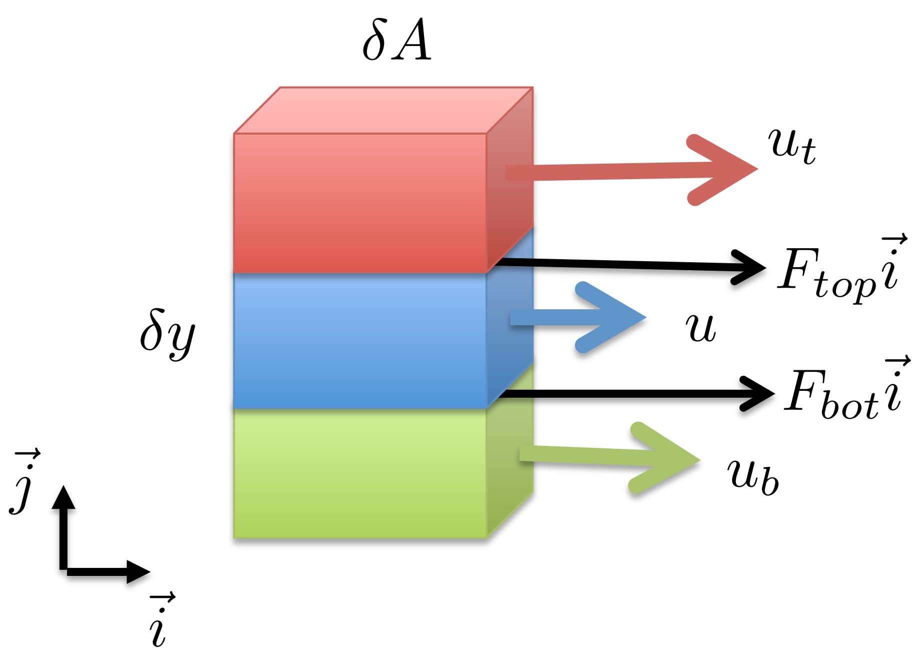
Where μ is the dynamic viscosity. Observe that this quantity is inversely proportional to the thickness of the fluid parcel δy. Why might this be the case?

Hence: $$F_{top} + F_{bot} = \frac{\mu \delta A}{\delta y}(u_t - 2u + u_b)$$

Viscous Force

The force per unit mass gives the acceleration:

$$\frac{F_{tot}}{m} = \frac{F_{tot}}{\rho \delta y \delta A} = \frac{\mu}{\rho} \frac{(u_t - 2u + u_b)}{\delta y^2}$$

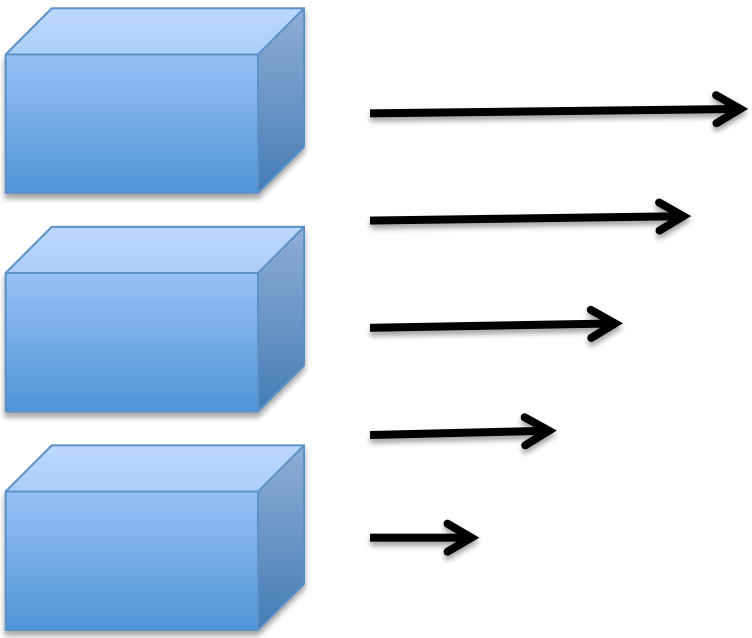


And so in the limit $\delta y \to 0$

$$\boxed{\frac{F_{tot}}{m} = \frac{\mu}{\rho} \left(\frac{\partial^2 u}{\partial y^2} \right)}$$

Viscous Force

Extending this derivation to each flow direction then yields the total acceleration due to viscosity.



Kinematic Viscosity $\nu = \frac{\mu}{\rho}$

$$\frac{\mathbf{F}_{visc}}{m} = \nu \nabla^2 \mathbf{u}$$

Note Laplacian (second derivative). What consequences does this have for linearly sheared flow?

Gravitational Force

Recall Newton's law of gravity: $\mathbf{F}_g = G\frac{Mm}{|\mathbf{r}|^2}\frac{\mathbf{r}}{|\mathbf{r}|}$

Gravitational constant:

$$G = 6.67384 \times 10^{-11}\ \mathrm{m}^3\ \mathrm{kg}^{-1}\ \mathrm{s}^{-2}$$

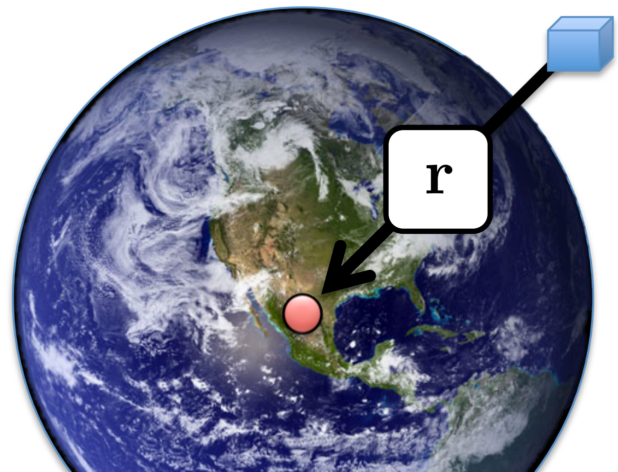
Mass of the Earth:

$$M = 5.972 \times 10^{24}\ \mathrm{kg}$$

But since the atmosphere is essentially a thin shell, we can make the approximation

$$|\mathbf{r}| \approx a \qquad a = 6.37122 \times 10^6\ \mathrm{m}$$

Define gravity at surface: $g = \frac{GM}{a^2}$



$$\frac{\mathbf{F}_g}{m} = -g\mathbf{k}$$

Total gravitational force acting on fluid parcel

Question: Can you calculate g from the above information?

Dynamical Equations of Motion (Cartesian, non-rotating fluid)

$$\frac{Du}{Dt} = \boxed{-\frac{1}{\rho}\frac{\partial p}{\partial x}} \boxed{+\nu\nabla^2 u}$$

$$\frac{Dv}{Dt} = \boxed{-\frac{1}{\rho}\frac{\partial p}{\partial y}} \boxed{+\nu\nabla^2 v}$$

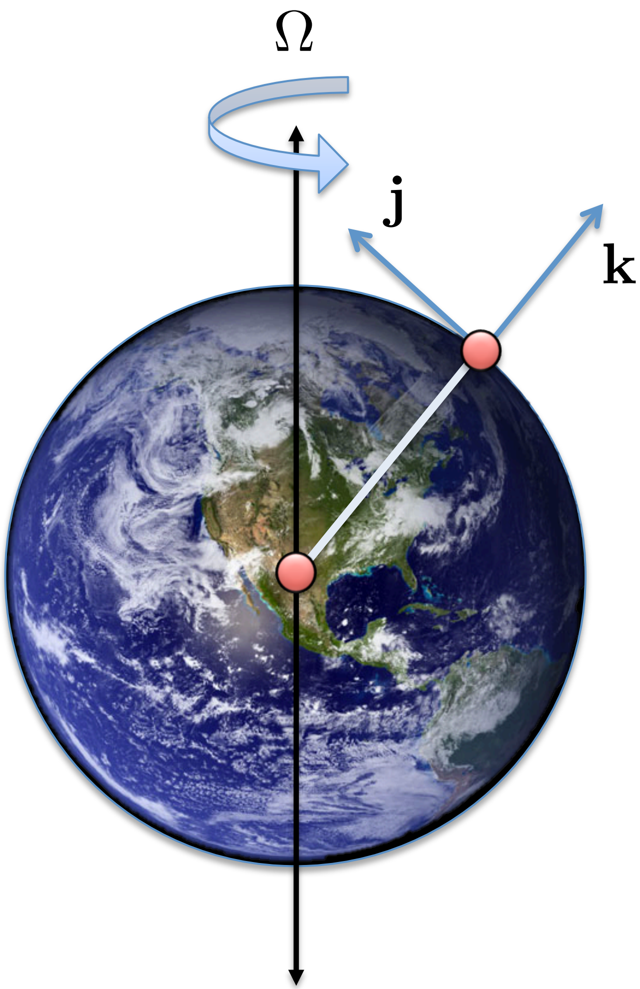
$$\frac{Dw}{Dt} = \boxed{-\frac{1}{\rho}\frac{\partial p}{\partial z}} \boxed{-g} \boxed{+\nu\nabla^2 w}$$

Pressure Gradient

Gravity

Viscosity

Coriolis / Centrifugal Force



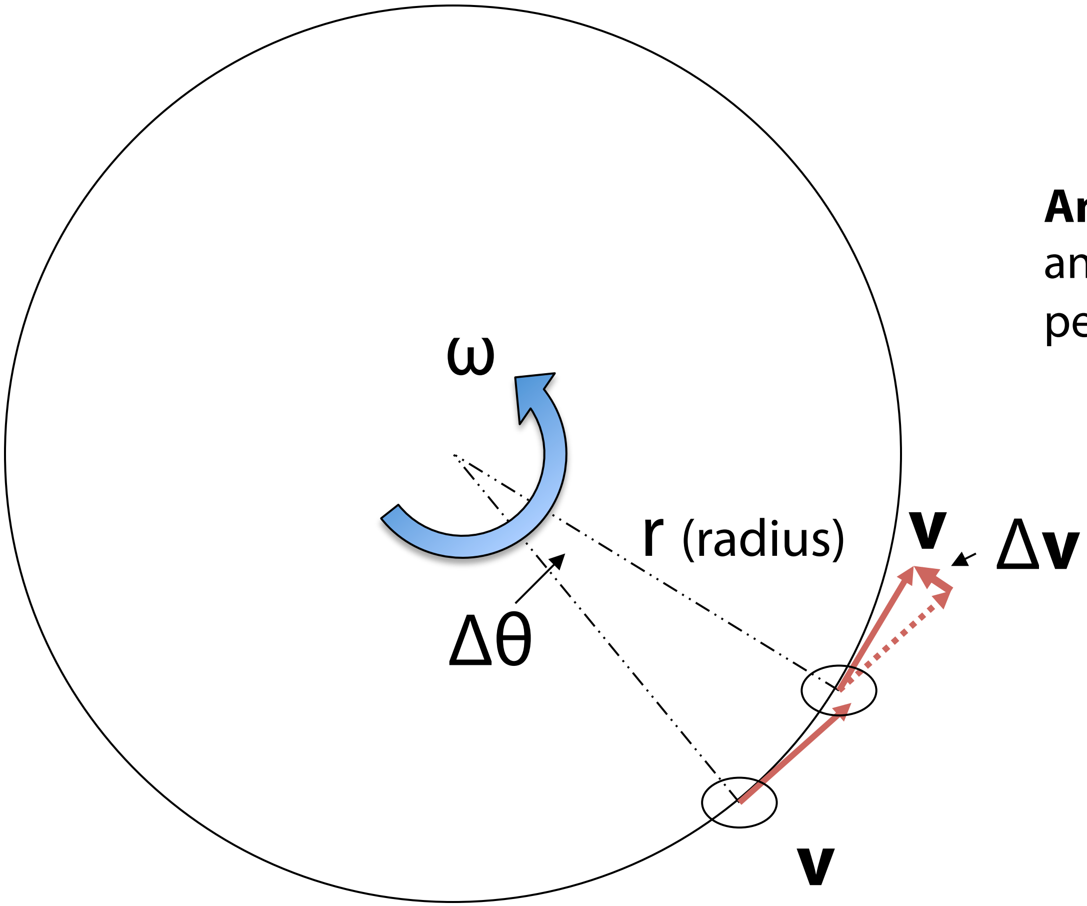
The Earth revolves around its axis at a certain rate Ω.

Coriolis and Centrifugal forces are known as ***apparent forces***, because they only exist because the reference frame is in motion.

The Coriolis force ***deflects*** fluid parcels as a consequence of the Earth's rotation.

The Centrifugal force attempts to push fluid parcels away from the axis of rotation.

Angular Momentum

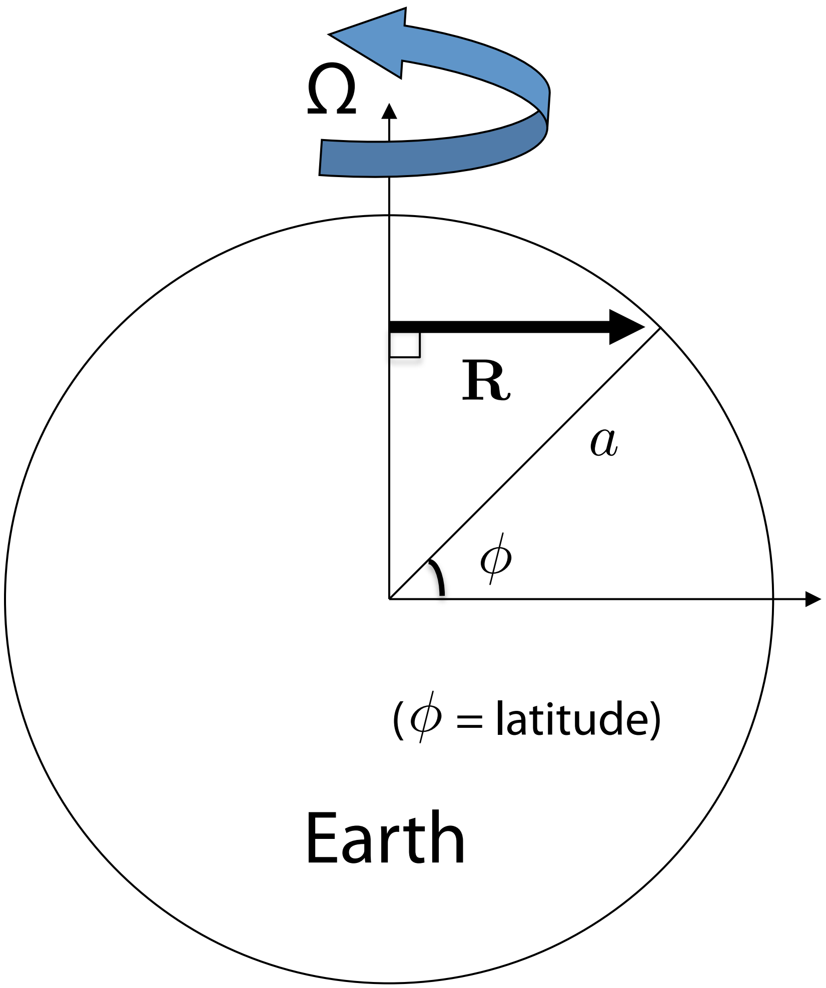


Angular velocity is defined as the angular frequency (in units of radians per second) multiplied by the radius:

$$|\mathbf{v}| = \omega r$$

Velocity vector

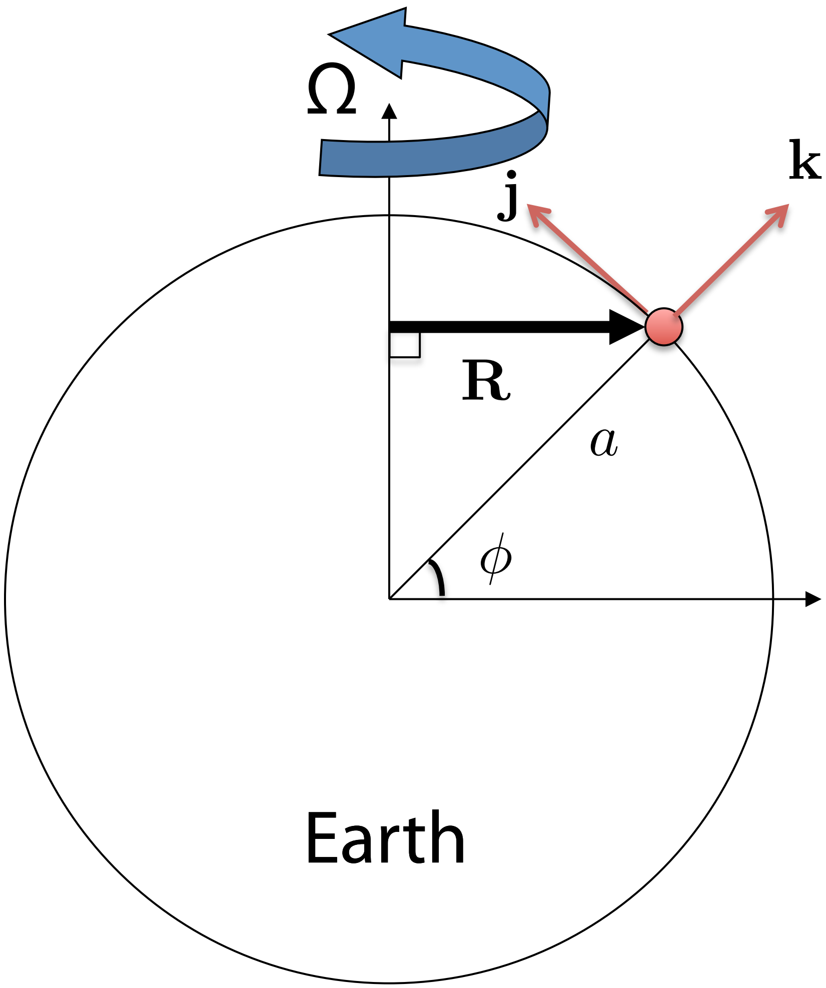
Angular Momentum



On the surface of the sphere the radius of rotation is equal to the perpendicular distance from the axis of rotation:

$$|R| = a\cos\phi$$

Angular Momentum



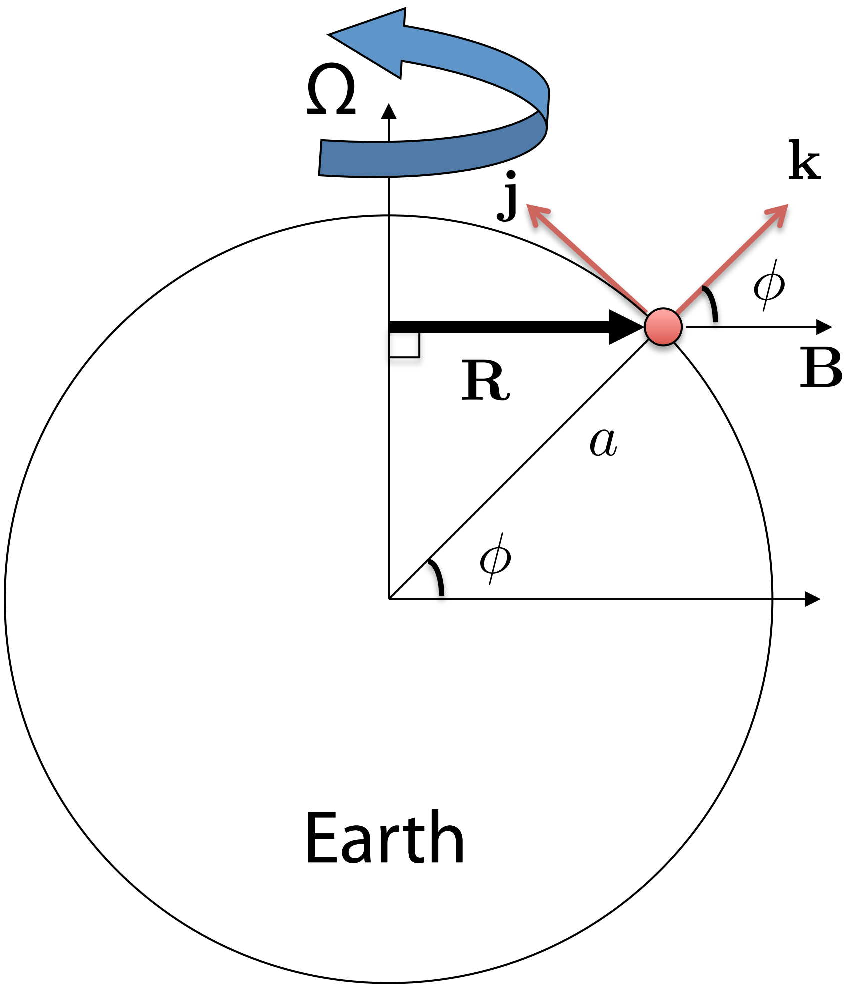
Define a local coordinate system:

$\mathbf{i}$ = Points towards the East (Zonal)
Longitudinal direction
(at red circle this is directed into the page)

$\mathbf{j}$ = Points towards the North (Meridional)
Latitudinal direction
(at red circle directed to top-left)

$\mathbf{k}$ = Points in the vertical
Local vertical coordinate
(at red circle directed to top-right)

Angular Momentum



Given a vector **B** in the same direction as **R**, whare the components of this vector in the local basis?

$$\mathbf{B} = B_y \mathbf{j} + B_z \mathbf{k}$$

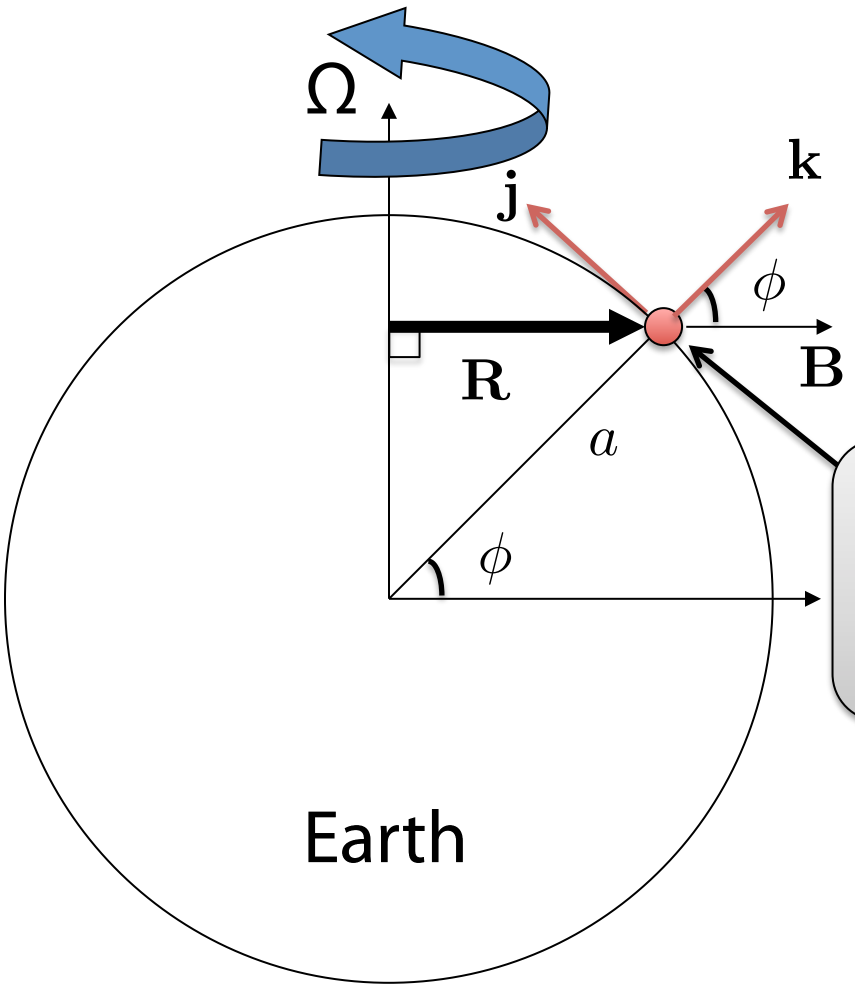
Verify:

$$B_y = -|\mathbf{B}| \sin\phi$$
$$B_z = \quad |\mathbf{B}| \cos\phi$$

Observe:

$$B_y^2 + B_z^2 = |\mathbf{B}|^2$$

Angular Momentum



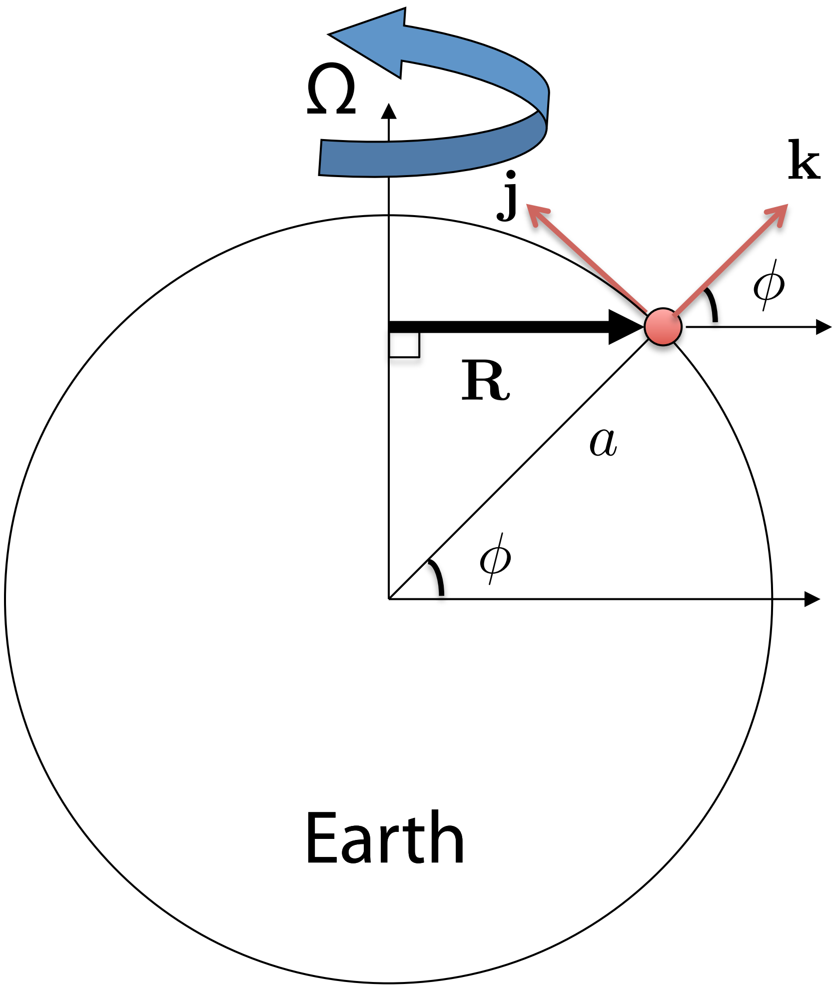
What is the velocity of this point **only** due to the rotation of the Earth?

Answer: Velocity is purely zonal

$$u_i = \Omega|\mathbf{R}|$$

$$\mathbf{u} = u_i\mathbf{i}$$

Angular Momentum



Velocity due to rotation of the Earth:

$$u_i = \Omega|\mathbf{R}|$$

Angular momentum vector **L** is equal to the cross product of the radial vector and the linear momentum

$$\mathbf{L} = \mathbf{R} \times m\mathbf{u}$$

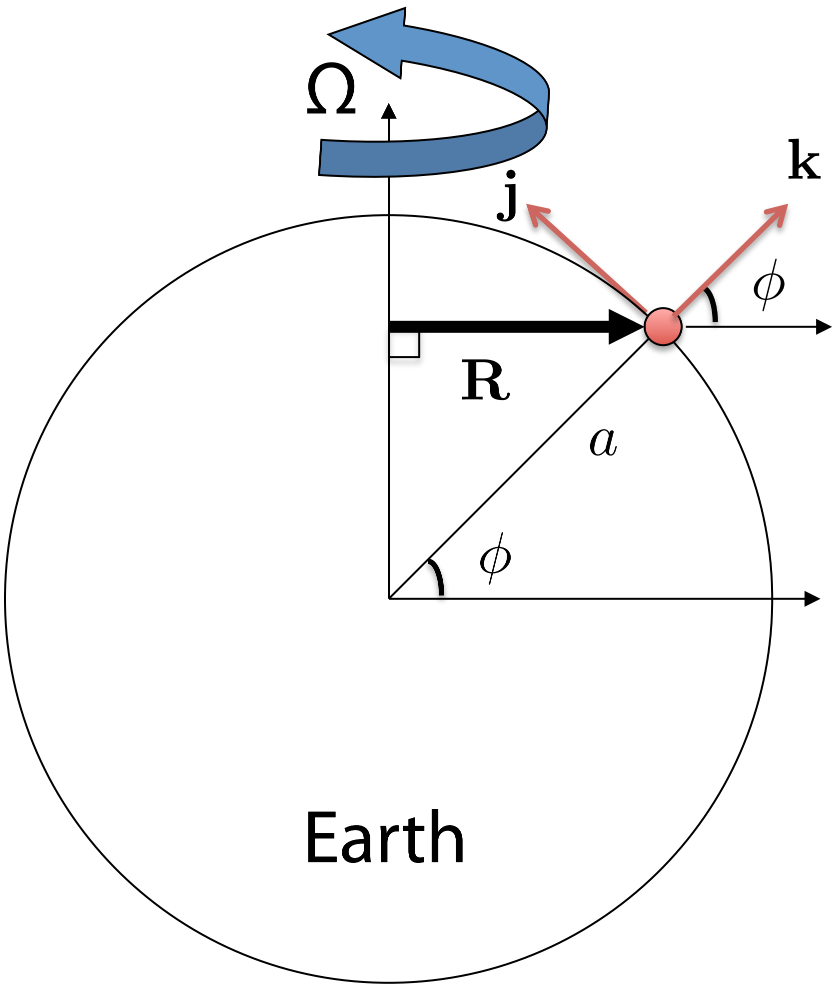
But **R** and **u** are perpendicular to one another, so

$$|\mathbf{L}| = |\mathbf{R}|mu_i$$

$$\Rightarrow \quad |\mathbf{L}| = m\Omega|\mathbf{R}|^2$$

The angular momentum vector points in the same direction as **Ω**.

Angular Momentum



Angular momentum due to rotation of the Earth:

$$|\mathbf{L}| = m\Omega|\mathbf{R}|^2$$

Rotation only

Write the radial vector in terms of a:

$$|\mathbf{R}| = a\cos\phi$$

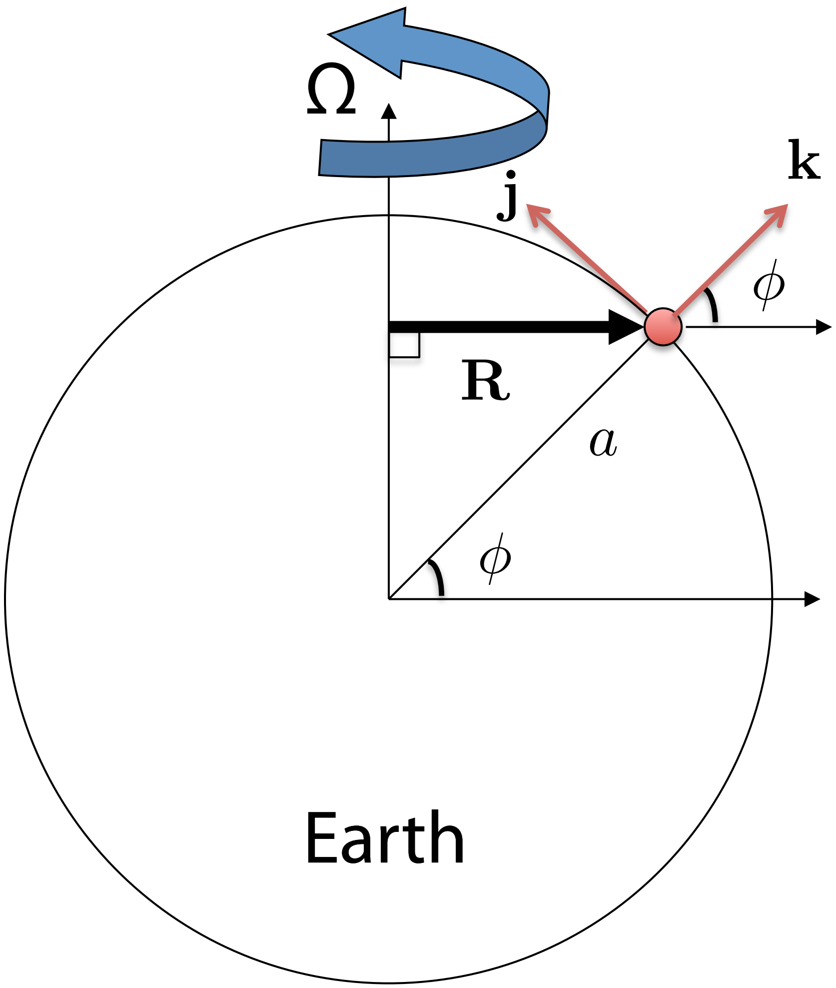
So for a fluid parcel at rest relative to the surface:

$$|\mathbf{L}| = m\Omega a^2\cos^2\phi$$

Angular momentum is relevant since it is a conserved quantity. That is,

$$\frac{D|\mathbf{L}|}{Dt} = 0$$

Angular Momentum



Assume the fluid parcel now possesses some zonal velocity u. The angular momentum due to its zonal velocity is

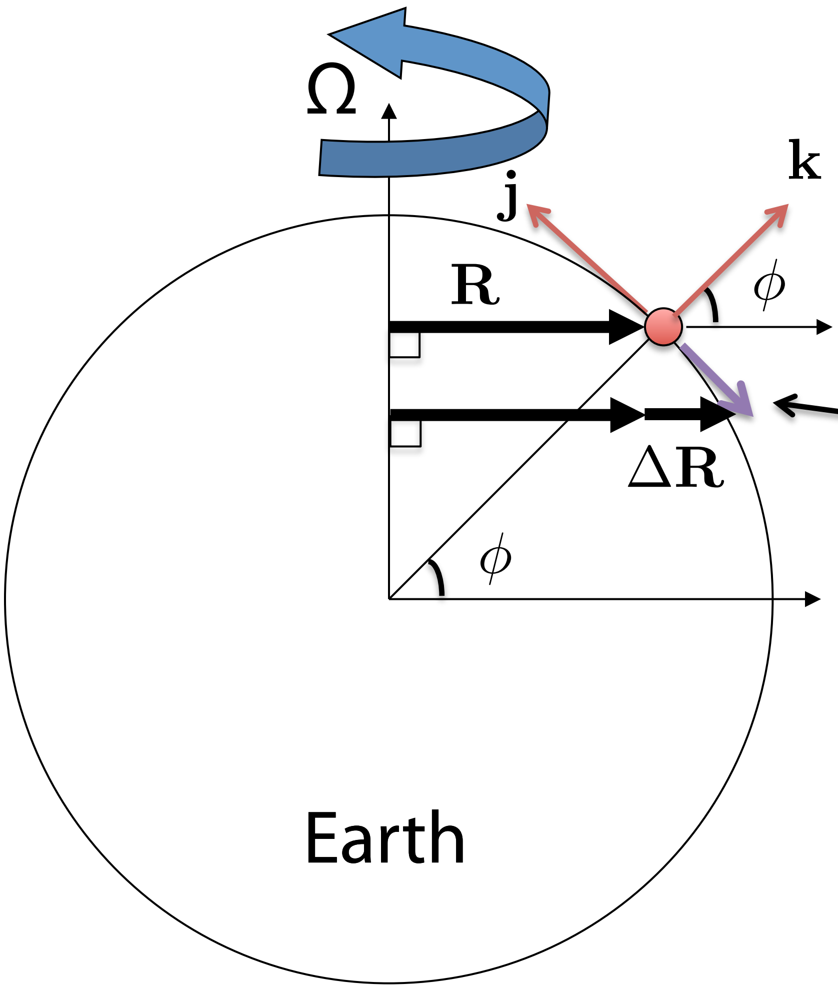
$$|\mathbf{L}| = mu|\mathbf{R}|$$

Zonal velocity only

Angular momentum including **both** rotation of the Earth and relative velocity u:

$$\boxed{\begin{aligned}\frac{|\mathbf{L}|}{m} &= \Omega|\mathbf{R}|^2 + u|\mathbf{R}| \\ &= |\mathbf{R}|^2\left(\Omega + \frac{u}{|\mathbf{R}|}\right)\end{aligned}}$$

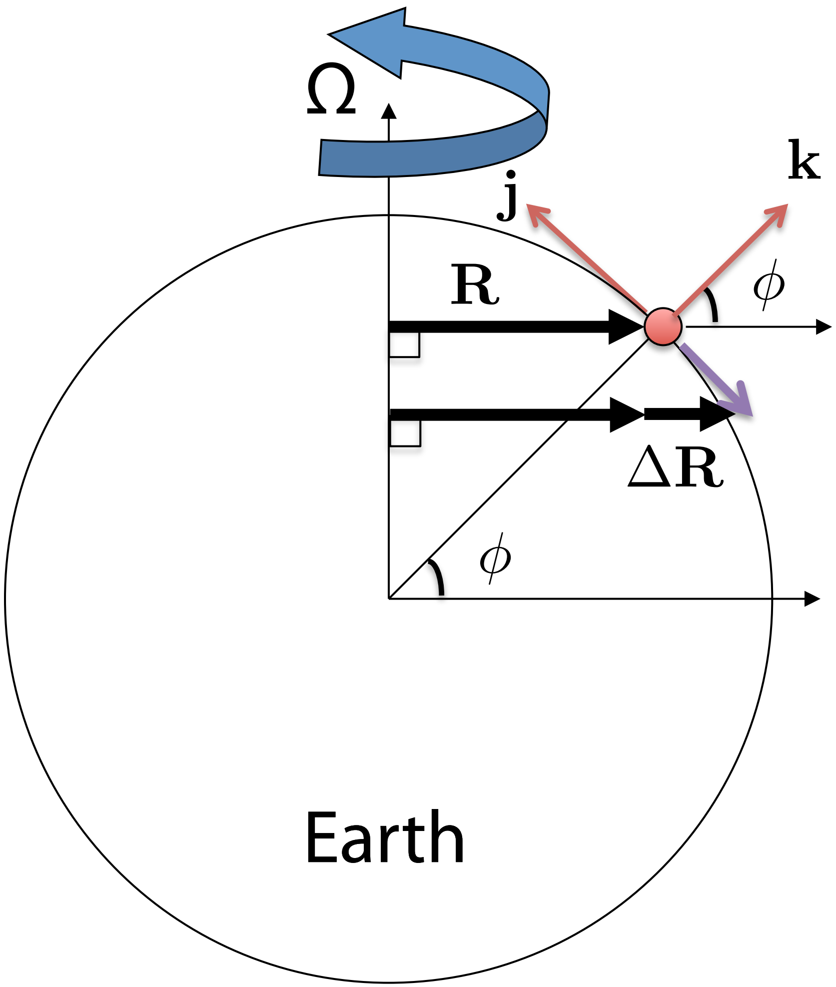
Meridional Displacement



Imagine the air parcel moves south (or north). What happens?

We get some change ΔR in the magnitude of **R**.

Meridional Displacement



Imagine the air parcel moves south (or north). What happens?

But if angular momentum is conserved, then u must change:

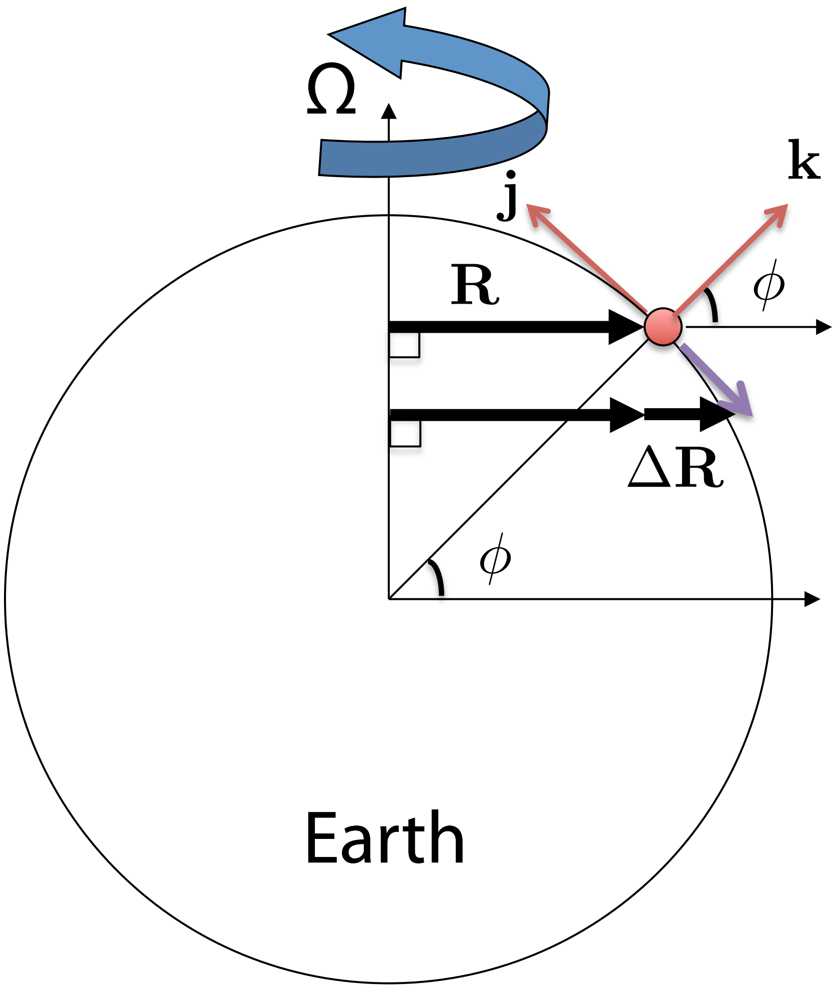
$$\frac{|\mathbf{L}|}{m} = |\mathbf{R}|^2 \left(\Omega + \frac{u}{|\mathbf{R}|} \right)$$

$$= (|\mathbf{R}| + \Delta R)^2 \left(\Omega + \frac{u + \Delta u}{|\mathbf{R}| + \Delta R} \right)$$

Solve (assuming small increments):

$$\Delta u \approx -2\Omega \Delta R - \frac{u}{|\mathbf{R}|} \Delta R$$

Meridional Displacement



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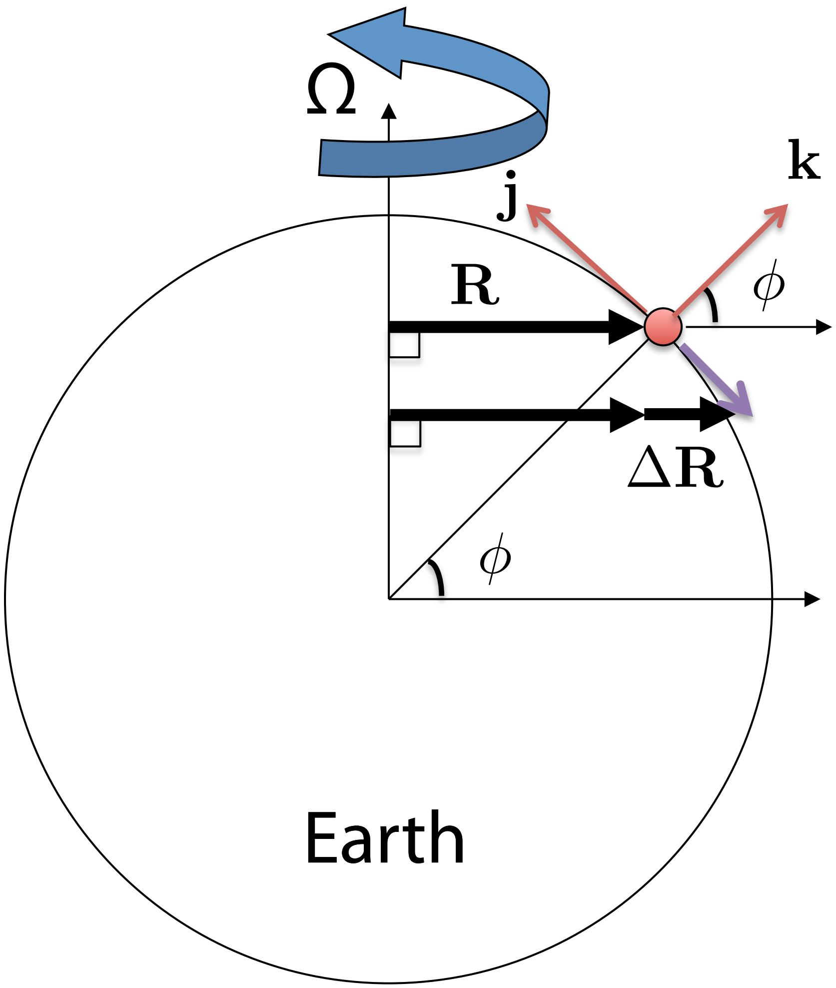
Meridional Displacement



Imagine the air parcel moves south (or north). What happens?

$$\Delta u \approx -2\Omega \Delta R - \frac{u}{|\mathbf{R}|}\Delta R$$

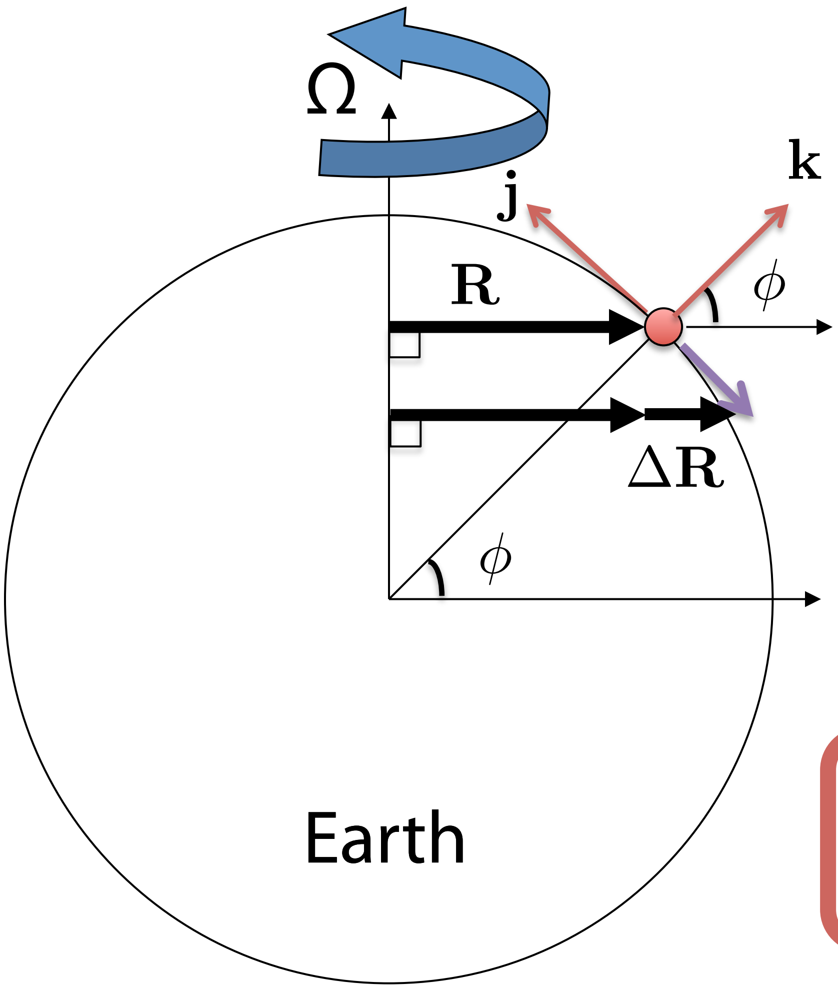
For southward displacement

$$\Delta R = -\sin\phi \Delta y \qquad (\Delta y = a\Delta\phi) \quad (\Delta y < 0)$$

Total change in velocity:

$$\Delta u \approx 2\Omega \sin\phi \Delta y + \frac{u}{a\cos\phi}\sin\phi\Delta y$$

Meridional Displacement



Imagine the air parcel moves south (or north). What happens?

$$\Delta u \approx -2\Omega \sin\phi \Delta y + \frac{u}{a\cos\phi} \sin\phi \Delta y$$

Divide through by Δt, take limit:

$$\frac{Du}{Dt} = -2\Omega \sin\phi \frac{Dy}{Dt} + \frac{u}{a} \tan\phi \frac{Dy}{Dt}$$

$$\boxed{\frac{Du}{Dt} = 2\Omega v \sin\phi + \frac{uv}{a} \tan\phi}$$

Coriolis Term

Geometric Term

Angular Momentum

Displacements of a fluid parcel in the meridional direction lead to the apparent forces:

$$\frac{Du}{Dt} = 2\Omega v \sin\phi + \frac{uv}{a}\tan\phi$$

What about if the fluid parcel is displaced zonally? *i.e.* if it experiences an acceleration

What about if the fluid parcel is displaced vertically?

Left as an **exercise** for the reader… Essentially requires repeating a similar analysis.

Then: Combine all three possibilities to obtain all Coriolis and geometric terms

Curvature / Coriolis in 3D

$$
\begin{aligned}
\frac{Du}{Dt} &= \boxed{2\Omega v \sin\phi} + \boxed{\frac{uv}{a}\tan\phi} - \boxed{2\Omega w \cos\phi} - \boxed{\frac{uw}{a}} \\
\frac{Dv}{Dt} &= \boxed{-2\Omega u \sin\phi} - \boxed{\frac{u^2}{a}\tan\phi - \frac{vw}{a}} \\
\frac{Dw}{Dt} &= \boxed{2\Omega u \cos\phi} + \boxed{\frac{u^2+v^2}{a}}
\end{aligned}
$$

Curvature

Coriolis

Coriolis Parameter

$$\frac{Du}{Dt} = 2\Omega v \sin\phi + \frac{uv}{a}\tan\phi - 2\Omega w \cos\phi - \frac{uw}{a}$$

$$\frac{Dv}{Dt} = -2\Omega u \sin\phi - \frac{u^2}{a}\tan\phi - \frac{vw}{a}$$

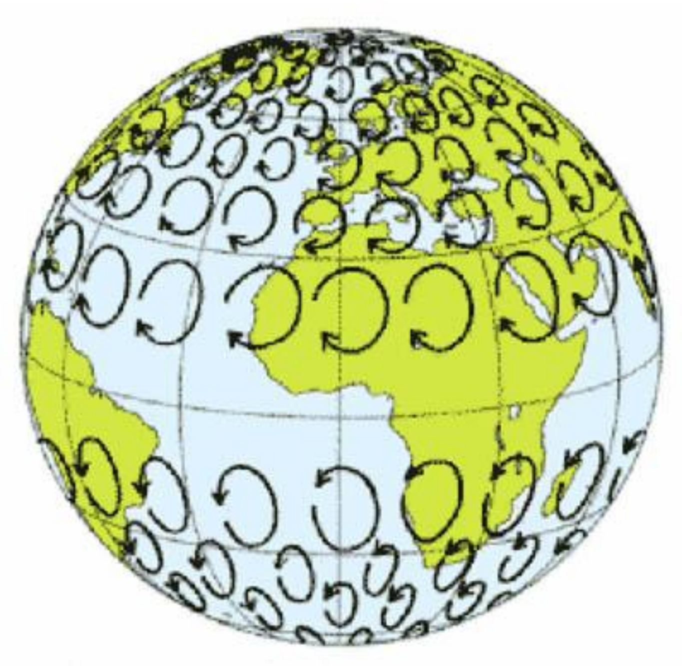
$$\frac{Dw}{Dt} = 2\Omega u \cos\phi + \frac{u^2 + v^2}{a}$$

Definition: The Coriolis Parameter f is defined as $f = 2\Omega \sin\phi$

For w small, the Coriolis force is then given by:

$$\frac{Du}{Dt} = fv \qquad \frac{Dv}{Dt} = -fu$$

Coriolis Force

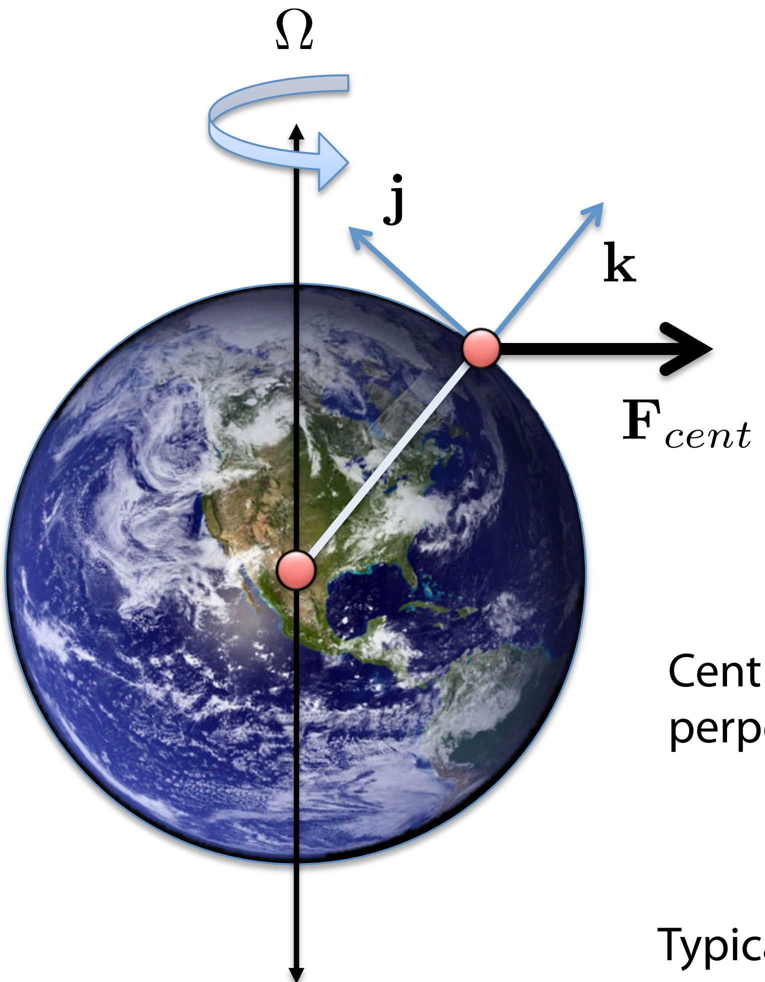


Deflection is to the right in the Northern hemisphere and to the left in the Southern hemisphere.

$$\frac{Du}{Dt} = fv \qquad \frac{Dv}{Dt} = -fu$$

Tendency is for fluid parcels to move in circles.

Centrifugal Force



Centrifugal force always works perpendicular to the axis of rotation:

$$\mathbf{F}_{cent} = \frac{mv^2}{r}\mathbf{R}$$

Typically small, so vertical component absorbed into the gravitational term and horizontal component absorbed into curvature.

Dynamic Equations of Motion (Spherical geometry, rotating fluid)

$$\frac{Du}{Dt} \boxed{- \frac{uv \tan\phi}{r} + \frac{uw}{r}} = \boxed{- \frac{1}{\rho r \cos\phi} \frac{\partial p}{\partial \lambda}} \boxed{+ 2\Omega v \sin\phi - 2\Omega w \cos\phi} \boxed{+ \nu \nabla^2 u}$$

$$\frac{Dv}{Dt} \boxed{+ \frac{u^2 \tan\phi}{r} + \frac{vw}{r}} = \boxed{- \frac{1}{\rho r} \frac{\partial p}{\partial \phi}} \boxed{- 2\Omega u \sin\phi} \boxed{+ \nu \nabla^2 v}$$

$$\frac{Dw}{Dt} \boxed{- \frac{u^2 + v^2}{r}} = \boxed{- \frac{1}{\rho} \frac{\partial p}{\partial r}} \boxed{- g} \boxed{+ 2\Omega u \cos\phi} \boxed{+ \nu \nabla^2 w}$$

Curvature

Pressure Gradient

Gravity

Coriolis

Viscosity