AOSS Mathematical Reference Sheets

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1 Trigonometry

Reciprocal properties

$$\sin x = \frac{1}{\csc\ x} \qquad \cos x = \frac{1}{\sec\ x}$$
$$\tan x = \frac{\sin x}{\cos x} \qquad \tan x = \frac{1}{\cot\ x}$$

Pythagorean identities

$$\begin{aligned}\sin^2 x + \cos^2 x &= 1\\ \sec^2 x - \tan^2 x &= 1\\ \csc^2 x - \cot^2 x &= 1\end{aligned}$$

Periodicity and Reflection

$$\sin\left(\tfrac{\pi}{2} - x\right) = \cos x \qquad \sin(-x) = -\sin(x)$$
$$\cos\left(\tfrac{\pi}{2} - x\right) = \sin x \qquad \cos(-x) = \cos(x)$$
$$\tan\left(\tfrac{\pi}{2} - x\right) = \cot x \qquad \tan(-x) = -\tan(x)$$

Complex notation

$$\begin{aligned}\sin x &= \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)\\ \cos x &= \frac{1}{2}\left(e^{ix} + e^{-ix}\right)\end{aligned}$$

Sum-Difference Formulas

$$\begin{aligned}\sin(a \pm b) &= \sin a \cos b \pm \cos a \sin b\\ \cos(a \pm b) &= \cos a \cos b \mp \sin a \sin b\\ \tan(a \pm b) &= \frac{\tan a \pm \tan b}{1 \mp \tan a \tan b}\end{aligned}$$

Double Angle Formulas

$$\begin{aligned}\sin(2x) &= 2 \sin x \cos x\\ \cos(2x) &= \cos^2 x - \sin^2 x\\ &= 2\cos^2 x - 1\\ &= 1 - 2\sin^2 x\\ \tan(2x) &= \frac{2\tan x}{1 - \tan^2 x}\end{aligned}$$

Sum-Product Formulas

$$\begin{aligned}\sin a + \sin b &= 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)\\ \sin a - \sin b &= 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)\\ \cos a + \cos b &= 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)\\ \cos a - \cos b &= -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)\end{aligned}$$

Product-Sum Formulas

$$\begin{aligned}\sin a \sin b &= \tfrac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]\\ \cos a \cos b &= \tfrac{1}{2}\left[\cos(a-b) + \cos(a+b)\right]\\ \sin a \cos b &= \tfrac{1}{2}\left[\sin(a+b) + \sin(a-b)\right]\\ \cos a \sin b &= \tfrac{1}{2}\left[\sin(a+b) - \sin(a-b)\right]\end{aligned}$$

2 Hyperbolic Trigonometry

Definition

$$\begin{aligned}\sinh x &= \tfrac{1}{2}\left(e^x - e^{-x}\right)\\ \cosh x &= \tfrac{1}{2}\left(e^x + e^{-x}\right)\\ \tanh x &= \frac{e^x - e^{-x}}{e^x + e^{-x}}\end{aligned}$$

Inverse Definition

$$\begin{aligned}\operatorname{arcsinh}(z) &= \ln(z + \sqrt{z^2+1})\\ \operatorname{arccosh}(z) &= \ln(z + \sqrt{z^2-1})\\ \operatorname{arctanh}(z) &= \tfrac{1}{2}\ln\left(\frac{1+z}{1-z}\right)\end{aligned}$$

Reflection

$$\begin{aligned}\sinh(-x) &= -\sinh(x)\\ \cosh(-x) &= \cosh(x)\\ \tanh(-x) &= -\tanh(x)\end{aligned}$$

Relation with Elliptic Trigonometry

$$\begin{aligned}\sinh x &= -i\sin(ix)\\ \cosh x &= \cos(ix)\\ \tanh x &= -i\tan(ix)\end{aligned}$$

Pythagorean Identities

$$\begin{aligned}\cosh^2 x - \sinh^2 x &= 1\\ \operatorname{sech}^2 x + \tanh^2 x &= 1\end{aligned}$$

Double Angle Formulas

$$\begin{aligned}\sinh(2x) &= 2\sinh x \cosh x\\ \cosh(2x) &= \cosh^2 x + \sinh^2 x\\ \tanh(2x) &= \frac{2\tanh x}{1 + \tanh^2 x}\end{aligned}$$

3 Single and Multivariable Calculus

First Fundamental Theorem of Calculus
Suppose f continuous and real valued on $[a, b]$, then

$$F(x) = \int_a^x f(t)dt \iff \frac{dF}{dx}(x) = f(x).$$

Second Fundamental Theorem of Calculus
Suppose f continuous and real valued on $[a, b]$ and let F satisfy $f(x) = \frac{dF}{dx}(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Chain Rule
If $f = f(y(x))$ then

$$\frac{df}{dx} = \frac{df}{dy}\frac{dy}{dx}$$

Liebniz (Product) Rule

$$\frac{d}{dx}(fg) = \frac{df}{dx}g + f\frac{dg}{dx}$$

Basic Integration by Parts

$$\int f dg = fg - \int g df$$

Multivariable Chain Rule
If $f = f(x_1(t), \ldots, x_n(t))$ then

$$\frac{df}{dt} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \cdots + \frac{\partial f}{\partial x_n}\frac{dx_n}{dt}$$

Jacobian

$$J(x_1, \ldots, x_n) = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_n} \end{vmatrix}$$

Integration by Parts
Suppose Ω is an open bounded subset of $\mathbf{R}^n$ with a piecewise smooth boundary $\partial\Omega$. If $u(x_1, \ldots, x_n)$ and $v(x_1, \ldots, x_n)$ are continuously differentiable on $\Omega \cup \partial\Omega$ then

$$\int_\Omega \frac{\partial u}{\partial x_i} v dx = \int_{\partial\Omega} uv\hat{\mathbf{n}}_i - \int_\Omega u\frac{\partial v}{\partial x_i}dx.$$

Change of Variables (1D)
Suppose f is a continuous function of y on some domain $S = [a, b]$ and $T = [\phi(a), \phi(b)]$ is the range $[a, b]$ under the differentiable mapping $y = \phi(x)$. Then

$$\int_T f(\phi(x))\frac{d\phi}{dx}dx = \int_S f(y)dy.$$

Change of Variables (2D)
Suppose f is a continuous function of (x, y) on some domain S and $T = \Psi(S)$ is an orientation-preserving diffeomorphism. Then

$$\int_T f(x, y)dxdy = \int_S f[x(u, v), y(u, v)] \left|\frac{\partial(x, y)}{\partial(u, v)}\right| dudv,$$

where

$$\left|\frac{\partial(x, y)}{\partial(u, v)}\right| = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix},$$

is the Jacobian in 2D.

Implicit Function Theorem
Given $F_i(u_1, \ldots, u_n, x_1, \ldots, x_n) = 0$ for $i = 1, \ldots, n$, then if the Jacobian satisfies

$$|JF(u_1, \ldots, u_n)| = \left|\frac{\partial(u_1, \ldots, u_n)}{\partial(x_1, \ldots, x_n)}\right| \neq 0,$$

then $u_1, \ldots, u_n$ can be solved for in terms of $x_1, \ldots, x_n$ and the partial derivatives of $u_1, \ldots, u_n$ can be determined by differentiating implicitly.

4 Series Expansions and Transforms

Taylor Series

$$f(x+\Delta x) = f(x) + \Delta x \frac{df}{dx}(x) + \tfrac{1}{2}\Delta x^2 \frac{d^2 f}{dx^2}(x) + \cdots + \frac{1}{n!}\Delta x^n \frac{d^n f}{dx^n}(x) + \cdots$$

Common Series Expansions

$$\begin{aligned}
e^x &= \sum_{i=0}^{\infty} \frac{1}{i!} x^i && = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \tfrac{1}{24}x^4 + O(x^5) \\
\sin(x) &= \sum_{i=0}^{\infty} (-1)^i \frac{1}{(2i+1)!} x^{2i+1} && = x - \tfrac{1}{6}x^3 + \tfrac{1}{120}x^5 - \tfrac{1}{5040}x^7 + O(x^9) \\
\cos(x) &= \sum_{i=0}^{\infty} (-1)^i \frac{1}{(2i)!} x^{2i} && = 1 - \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4 - \tfrac{1}{720}x^6 + O(x^8) \\
\sinh(x) &= \sum_{i=0}^{\infty} \frac{1}{(2i+1)!} x^{2i+1} && = x + \tfrac{1}{6}x^3 + \tfrac{1}{120}x^5 + \tfrac{1}{5040}x^7 + O(x^9) \\
\cosh(x) &= \sum_{i=0}^{\infty} \frac{1}{(2i)!} x^{2i} && = 1 + \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4 + \tfrac{1}{720}x^6 + O(x^8) \\
\frac{1}{1-x} &= \sum_{i=0}^{\infty} x^i && = 1 + x + x^2 + x^3 + x^4 + O(x^5) \\
\ln(1+x) &= \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} x^i && = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + O(x^5) \\
\sqrt{1+x} &= \sum_{i=1}^{\infty} \frac{(-1)^i}{i!} \left(-\tfrac{1}{2}\right) \cdots \left(-\tfrac{1}{2} - i\right) x^i && = 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3 - \tfrac{5}{128}x^4 + O(x^5)
\end{aligned}$$

Arithmatic and Geometric Series

$$\sum_{k=0}^{n} k = \frac{n(n-1)}{2}, \qquad \sum_{k=0}^{n} r^k = \frac{1-r^n}{1-r}.$$

Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}, \qquad \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Fourier Series

For a periodic function with period L, we have

$$\begin{aligned}
f(x) &= a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} \right), \\
a_n &= \frac{1}{L}\int_{-L}^{L} f(x) \cos\frac{n\pi x}{L} dx, \\
b_n &= \frac{1}{L}\int_{-L}^{L} f(x) \sin\frac{n\pi x}{L} dx.
\end{aligned}$$

Complex Form of the Fourier Series

$$\begin{aligned}
f(x) &= \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{in\pi x}{L}\right), \\
c_n &= \frac{1}{2L}\int_{-L}^{L} f(x) \exp\left(-\frac{in\pi x}{L}\right) dx, \\
&= \begin{cases} \tfrac{1}{2}(a_n - ib_n), & \text{when n ¿ 0,} \\ a_0 & \text{when n = 0,} \\ \tfrac{1}{2}(a_{-n} + ib_{-n}), & \text{when n ¡ 0.} \end{cases}
\end{aligned}$$

Fourier Transform and Inverse Transform

$$\begin{aligned}
\mathcal{F}(\omega) &= \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \\
f(t) &= \frac{1}{2\pi}\int_{\infty}^{\infty} \mathcal{F}(\omega) e^{i\omega x} d\omega
\end{aligned}$$

Laplace Transform

$$\mathcal{F}(s) = \mathcal{L}\left[f(t)\right] = \int_{0^-}^{+\infty} f(t) e^{-st} dt.$$

5 Vector Calculus

Vector Identities

$$\begin{aligned}\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) &= \mathbf{B}(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}\cdot\mathbf{B})\\ \mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) &= \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})=\mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})\end{aligned}$$

Zero Identities

$$\begin{aligned}\nabla\times(\nabla\psi) &= 0\\ \nabla\cdot(\nabla\times\mathbf{A}) &= 0\end{aligned}$$

Laplacian of a scalar

$$\nabla^2\psi = \nabla\cdot(\nabla\psi)$$

Curl of a curl

$$\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}$$

Distributive property

$$\begin{aligned}\nabla\cdot(\mathbf{A}+\mathbf{B}) &= \nabla\cdot\mathbf{A}+\nabla\cdot\mathbf{B}\\ \nabla\times(\mathbf{A}+\mathbf{B}) &= \nabla\times\mathbf{A}+\nabla\times\mathbf{B}\end{aligned}$$

Product rules for a scalar field

$$\begin{aligned}\nabla(\psi\phi) &= \phi\nabla\psi+\psi\nabla\phi\\ \nabla\cdot(\psi\mathbf{A}) &= \mathbf{A}\cdot\nabla\psi+\psi\nabla\cdot\mathbf{A}\\ \nabla\times(\psi\mathbf{A}) &= \psi(\nabla\times\mathbf{A})-\mathbf{A}\times\nabla\psi\end{aligned}$$

Gradient of a square

$$\tfrac{1}{2}\nabla\mathbf{A}^2 = \tfrac{1}{2}\nabla(\mathbf{A}\cdot\mathbf{A}) = \mathbf{A}\times(\nabla\times\mathbf{A})+(\mathbf{A}\cdot\nabla)\mathbf{A}.$$

Product rule for a vector dot product

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{A}\cdot\nabla)B+(\mathbf{B}\cdot\nabla)A+\mathbf{A}\times(\nabla\times\mathbf{B})+\mathbf{B}\times(\nabla\times\mathbf{A})$$

Product rule for a vector cross product

$$\begin{aligned}\nabla\cdot(\mathbf{A}\times\mathbf{B}) &= \mathbf{B}\cdot(\nabla\times\mathbf{A})-\mathbf{A}\cdot(\nabla\times\mathbf{B})\\ \nabla\times(\mathbf{A}\times\mathbf{B}) &= \mathbf{A}(\nabla\cdot\mathbf{B})-\mathbf{B}(\nabla\cdot\mathbf{A})+(\mathbf{B}\cdot\nabla)\mathbf{A}-(\mathbf{A}\cdot\nabla)\mathbf{B}\end{aligned}$$

Green's theorem
Let $\mathcal{C}$ be a positively oriented, piecewise smooth, simple closed curve in the plane $\mathbf{R}^2$, and let $\mathcal{D}$ be the region bounded by $\mathcal{C}$. If $L(x,y)$ and $M(x,y)$ are functions defined on an open region containing $\mathcal{D}$ and have continuous partial derivatives there, then

$$\oint_{\mathcal{C}} Ldx+Mdy = \iint_{\mathcal{D}}\left(\frac{\partial M}{\partial x}-\frac{\partial L}{\partial y}\right)dA.$$

Gauss' divergence theorem
Suppose $\mathcal{V}$ is a subset of $\mathbf{R}^n$ (for $n=3$, $\mathcal{V}$ represents a volume in 3D space) which is compact and has a piecewise smooth boundary $\partial\mathcal{V}$. If $\mathbf{F}$ is a continuously differentiable vector field on some neighborhood of $\mathcal{V}$, then

$$\int_{\mathcal{V}}\nabla\cdot\mathbf{F}dV = \oint_{\partial\mathcal{V}}\mathbf{F}\cdot\hat{\mathbf{n}}dS.$$

Stokes' curl theorem
Suppose $\mathcal{S}$ is an oriented, compact surface with boundary $\partial\mathcal{S}$ in $\mathbf{R}^3$. Suppose $\mathbf{F}$ is a continuously differentiable vector field on some simply connected neighborhood of $\mathcal{S}\cup\partial\mathcal{S}$, then

$$\int_{\mathcal{S}}(\nabla\times\mathbf{F})\cdot d\mathbf{S} = \int_{\partial\mathcal{S}}\mathbf{F}\cdot d\ell.$$

Integration by Parts
Suppose Ω is an open bounded subset of $\mathbf{R}^n$ with a piecwise smooth boundary $\partial\Omega$. If scalar field u and vector field $\mathbf{v}$ are continuously differentiable on $\Omega\cup\partial\Omega$ then

$$\int_{\Omega}\nabla u\cdot\mathbf{v}dV = \int_{\partial\Omega}u\mathbf{v}\cdot\hat{\mathbf{n}}dS-\int_{\Omega}u\nabla\cdot\mathbf{v}dV.$$

6 Thermodynamics

Gas constant

$$R = 8.314\ 472\ J\ kg^{-1}\ K^{-1}$$

First Law of Thermodynamics

$$dU = \delta Q - \delta W = \delta Q - pdV.$$

Second Law of Thermodynamics

$$dQ = TdS.$$

Ideal Gas Law

$$pV = nRT$$

Heat capacity (const pressure)

$$\begin{aligned} C_p &= \left(\frac{\partial Q}{\partial T}\right)_p = T\left(\frac{\partial S}{\partial T}\right)_p \\ &= \left(\frac{\partial U}{\partial T}\right)_p + p\left(\frac{\partial V}{\partial T}\right)_p. \end{aligned}$$

Heat capacity at (const volume)

$$\begin{aligned} C_V &= \left(\frac{\partial Q}{\partial T}\right)_V = T\left(\frac{\partial S}{\partial T}\right)_V \\ &= \left(\frac{\partial U}{\partial T}\right)_V. \end{aligned}$$

Ratio of specific heats

$$\gamma = \frac{C_p}{C_V}$$

Enthalpy

$$H \equiv U + pV = \mu N + TS.$$

Helmholtz Free Energy

$$A \equiv U - TS = \mu N - pV.$$

Gibbs Free Energy

$$\begin{aligned} G &\equiv U + pV - TS \\ &= H - TS = \mu N. \end{aligned}$$

Maxwell Relations

$$\begin{aligned} \left(\frac{\partial T}{\partial V}\right)_{S,N} &= -\left(\frac{\partial p}{\partial S}\right)_{V,N} \\ \left(\frac{\partial T}{\partial V}\right)_{p,N} &= -\left(\frac{\partial p}{\partial S}\right)_{T,N} \\ \left(\frac{\partial T}{\partial p}\right)_{S,N} &= \left(\frac{\partial V}{\partial S}\right)_{p,N} \\ \left(\frac{\partial T}{\partial p}\right)_{V,N} &= \left(\frac{\partial V}{\partial S}\right)_{T,N} \end{aligned}$$

Additional Relations

$$\left(\frac{\partial T}{\partial S}\right)_V = \frac{T}{C_v} \qquad \left(\frac{\partial T}{\partial S}\right)_p = \frac{T}{C_p}$$

$$\left(\frac{\partial S}{\partial U}\right)_{V,N} = \frac{1}{T} \qquad \left(\frac{\partial S}{\partial V}\right)_{N,U} = \frac{p}{T}$$

Thermodynamic Efficiency (for a heat engine)

$$\eta \equiv \frac{\text{work}}{\text{heat in}}.$$

Polytropic Gas Law (isentropic process)

$$\frac{p}{\rho^\gamma} = \text{const.}$$

Isobaric Processes

$$dp = 0, \quad \delta Q = C_p dT = \gamma dU$$

Isothermal processes

$$dT = 0, \quad \delta Q = -V dp = pdV = \delta W$$

Isochoric (incompressible) processes

$$dV = 0, \quad \delta Q = C_V dT = dU$$

Adiabatic processes

$$\delta Q = 0, \quad C_p dT = V dp, \quad C_V dT = -pdV$$

7 Fluid Mechanics

Material derivative

$$\frac{D\mathbf{X}}{Dt} = \frac{\partial \mathbf{X}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{X}$$

Continuity equation

$$\begin{aligned}\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) &= 0\\ \frac{D\rho}{Dt} + \rho(\nabla\cdot\mathbf{v}) &= 0\end{aligned}$$

Euler Fluids (adiabatic)

$$\rho\frac{Du_i}{Dt} = -\nabla_i p, \qquad \frac{Ds}{Dt} = 0.$$

Steady incompressible Bernoulli equation

$$\tfrac{1}{2}u^2 + w + gz = \text{constant along streamline}$$

Time-dependent irrotational Bernoulli equation (potential flow)

$$\frac{\partial\phi}{\partial t} + \tfrac{1}{2}u^2 + w + gz = 0$$

Circulation

$$\Gamma = \oint_c \mathbf{v}\cdot d\ell$$

Navier-Stokes equations (incompressible)

$$\frac{\partial}{\partial t}(\rho u_i) + \nabla_j(\rho u_i u_j) = \rho\frac{Du_i}{Dt} = \underbrace{-\nabla_i p}_{\text{pressure}} + \underbrace{\rho g_i}_{\text{gravity}} + \underbrace{\nu\nabla^2 u_i}_{\text{viscosity}} + \underbrace{\frac{\partial\tau_{ij}}{\partial x_j}}_{\text{body forces}}$$

Energy conservation equation

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho u^2 + \rho\epsilon\right) + \nabla\cdot\left[\rho\mathbf{u}\left(\tfrac{1}{2}u^2 + \epsilon + \frac{p}{\rho}\right)\right] = \underbrace{\rho\mathbf{g}\cdot\mathbf{u}}_{\text{gravity}} + \underbrace{\frac{\partial}{\partial x_j}(u_i\tau_{ij})}_{\text{surface forces}} + \underbrace{q}_{\text{heating}}$$

Mechanical energy equation

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho u^2\right) + \nabla\cdot\left(\rho\mathbf{u}\cdot\tfrac{1}{2}u^2\right) = \underbrace{-\mathbf{u}\cdot\nabla p}_{\text{pressure}} + \underbrace{\rho\mathbf{g}\cdot\mathbf{u}}_{\text{gravity}} + \underbrace{\frac{\partial}{\partial x_j}(u_i\tau_{ij})}_{\text{surface forces}} \underbrace{-\tfrac{1}{2}\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)^2}_{\text{viscous heating}}$$

Thermal energy equation

$$\frac{\partial}{\partial t}(\rho\epsilon) + \nabla\cdot(\mathbf{u}\rho\epsilon) = \underbrace{-p(\nabla\cdot\mathbf{u})}_{\text{expansion/compression}} + \underbrace{q}_{\text{heating}} \underbrace{+\tfrac{1}{2}\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)^2}_{\text{viscous heating}}$$

Nondimensional Constants

$Ra \equiv \frac{g\alpha(T - T_\infty)d^3}{\nu\kappa}$	(Rayleigh)	$Re \equiv \frac{UL}{\nu}$	(Reynolds)
$Pr \equiv \frac{\nu}{\kappa}$	(Prandtl)	$Ri \equiv \frac{gh}{U^2}$	(Richardson)
$Fr \equiv \frac{U}{\sqrt{gH}}$	(Froude)	$Ro \equiv \frac{U}{Lf}$	(Rossby)
$Ta \equiv \frac{4\Omega^2 R^2}{\nu^2}$	(Taylor)	$M \equiv \frac{v_0}{v_s}$	(Mach)

8 Clouds and Precipitation

Constants for dry air (near $0\,^\circ C$)

$$\begin{aligned}\rho_a &= 28.96 g\ mol^{-1}\\ R_a &= 287 J\ kg^{-1}\ K^{-1}\\ c_{pa} &= 1005 J\ kg^{-1}\ K^{-1}\\ c_{va} &= 718 J\ kg^{-1}\ K^{-1}\\ \kappa &= R_a/c_{pa} = 0.286\end{aligned}$$

Constants for water vapor (near $0\,^\circ C$)

$$\begin{aligned}\rho_v &= 18.01 g\ mol^{-1}\\ R_v &= 461.5 J\ kg^{-1}\ K^{-1}\\ c_{pv} &= 1870 J\ kg^{-1}\ K^{-1}\\ c_{vv} &= 1410 J\ kg^{-1}\ K^{-1}\\ L_f &= 3.340\times 10^5 J\ kg^{-1}\\ L_v &= 2.260\times 10^6 J\ kg^{-1}\end{aligned}$$

Ratio of Specific Gas Constants

$$\epsilon = R_a/R_v = 0.622$$

Potential Temperature

$$\theta = T\left(\frac{100kPa}{p}\right)^{\kappa}$$

Clausius-Clapeyron Equation

$$\frac{de_s}{dT} = \frac{S_2 - S_1}{V_2 - V_1} = \frac{L_v e_s}{R_v T^2}$$

Clausius-Clapeyron Equation (Bolton)
For T in degrees C and e_s in mbar,

$$e_s = 6.112 \exp\left[\frac{17.67\cdot T}{T + 243.5}\right],$$

Water Vapor Partial Pressure

$$e = \rho_v R_v T = \rho_v \frac{R_a}{\epsilon} T$$

Mixing Ratio

$$w \equiv \frac{m_v}{m_d} = \frac{\rho_v}{\rho_d} = \epsilon\frac{e}{p-e}$$

Specific Humidity

$$q \equiv \frac{m_v}{m_d + m_v} = \frac{\rho_v}{\rho_d + \rho_v} = \epsilon\frac{e}{p - (1-\epsilon)e}$$

Virtual Temperature

$$T_v = T\left[\frac{1 + w/e}{1 + w}\right]$$

Equivalent Temperature

$$T_e = T + \frac{L_v}{c_p} q$$

Equivalent Potential Temperature

$$\theta_e \approx T\left(\frac{p_0}{p}\right)^{\kappa} \exp\left(\frac{L_v q}{c_p T_{LCL}}\right)$$

Pseudoadiabatic Temperature Lapse Rate

$$\frac{dT}{dz} = \frac{\Gamma_d\left(1 + \frac{L_v q_s}{R_d T}\right)}{\left(1 + \frac{L_v^2 q_s}{c_p R_v T^2}\right)}$$

Scale Height

$$p = p_0 \exp\left(-\frac{1}{H}(z - z_0)\right)$$

Energy Equation

$$d\left(\tfrac{1}{2}u^2 + gz\right) + \alpha dp = -\mathbf{f}\cdot d\ell$$

Pressure Change for a Convective Vortex

$$\Delta p = p_\infty\left[1 - \exp\left(\frac{\gamma\eta(c_p\Delta T + L_v\Delta q)}{(\gamma\eta - 1)RT}\right)\right]$$

9 Radiative Transfer

Planck Constant

$$h = 6.626 \times 10^{-34} J\ sec$$

Boltzmann Constant

$$k = 1.3806 \times 10^{-23} J\ K^{-1}$$

Monochromatic Intensity

$$I_\lambda = \frac{dE_\lambda}{\cos\theta d\Omega d\lambda dt dA}$$

Monochromatic Flux Density

$$F_\lambda = \int_\Omega I_\lambda \cos\theta d\Omega = \int_0^{2\pi}\int_0^{\pi/2} I_\lambda(\theta,\phi)\cos\theta\sin\theta d\theta d\phi$$

Monochromatic Flux Density (Isotropic)

$$F_\lambda = \pi I_\lambda$$

Absorptivity

$$A(\nu) = \frac{I_{\nu,abs}(\nu)}{I_{\nu,incident}(\nu)}$$

Emissivity

$$\epsilon(\nu) = \frac{\text{Emission}}{\text{Blackbody Emission}}$$

Transmissivity

$$\mathcal{T}_v = \exp(-\tau(\nu))$$

Planck's Blackbody Radiation Law

$$B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT}-1}$$

$$B_\lambda(T) = \frac{2hc^2}{\lambda^5}\frac{1}{e^{hc/k\lambda T}-1}$$

Stefan-Boltzmann Law

$$F = \sigma T_e^4$$

Wien's Displacement Law

$$T/\nu_{max} = \text{const} \qquad T\lambda_{max} = \text{const}$$

Pressure Broadening

$$k_\nu = \frac{S}{\pi}\frac{\alpha}{(\nu-\nu_0)^2+\alpha^2} = Sf(\nu-\nu_0)$$

$$\alpha = \alpha_0(p/p_0)(T_0/T)^n$$

Doppler Broadening

$$k_\nu = \frac{S}{\alpha_D\sqrt{\pi}}\exp\left[-\left(\frac{\nu-\nu_0}{\alpha_D}\right)^2\right]$$

$$\alpha_D = \nu_0(2KT/mc^2)^{1/2}$$

Optical Depth

$$\chi = \int_{s_1}^{s_2}\rho(s)k_m(\nu,s)ds, \quad \tau = \chi\cos\theta$$

Beer's Law

$$dI_\lambda \propto -I_\lambda ds$$

General Radiative Transfer Equation

$$\frac{dI_\lambda}{d\tau} = I_\lambda - J_\lambda$$

Radiative Transfer Equation

$$\frac{dI_\lambda}{d\tau} = I_\lambda(\Omega) - (1-\omega)B_\lambda(T) - \frac{\omega}{4\pi}\int_{4\pi} I_\lambda(\Omega')P_\lambda(\Omega,\Omega')d\Omega'.$$

Asymmetry Factor

$$g = \frac{1}{2}\int_{-1}^{1} P(\mu)\mu d\mu.$$

10 Atmospheric Dynamics

Constants

$$\begin{aligned} g &= 9.80616 m/s^2 \\ R_d &= 287 J\ kg^{-1}\ K^{-1} \\ c_v &= 717 J\ kg^{-1}\ K^{-1} \\ c_p &= 1004 J\ kg^{-1}\ K^{-1} \\ a &= 6.371 km \\ \Omega &= 7.292\times 10^{-5} s^{-1} \\ \kappa &= R_d/c_p \approx 0.286 \end{aligned}$$

Coriolis Parameter

$$f = 2\Omega \sin\theta$$

Beta-plane Approximation

$$f = f_0 + \beta y, \qquad f_0 = 2\Omega\sin\theta_0, \qquad \beta = \frac{2\Omega\cos\theta_0}{a}$$

Hydrostatic Equation

$$dp = -\rho g dz$$

Dry Adiabatic Lapse Rate

$$\Gamma_d = -\frac{\partial T}{\partial z} = \frac{g}{c_p}$$

Brunt-Väisälä Frequency

$$N^2 = \frac{g}{\Theta}\frac{\partial \Theta}{\partial z}$$

Static Stability Parameter

$$S_p = \left(\frac{R_d T}{p c_p} - \frac{\partial T}{\partial p}\right) = -\frac{T}{\Theta}\frac{\partial\Theta}{\partial p} = \frac{\Gamma_d - \Gamma}{\rho g}$$

Hypsometric Equation

$$z_T = (z_2 - z_1) = \frac{R_d}{g}\int_{p_2}^{p_1}\frac{T}{p}dp$$

Thermal Wind (Pressure Coordinates)

$$\begin{aligned} \mathbf{v}_T = \mathbf{v}_g(p_2) - \mathbf{v}_g(p_1) &= \frac{R_d}{f}\mathbf{k}\times\nabla_p\langle T\rangle \ln\left(\frac{p_1}{p_2}\right) \\ &= \frac{1}{f}\mathbf{k}\times\nabla_p(\Phi_2 - \Phi_1) \end{aligned}$$

Conservation of PV (Barotropic Fluid)

$$\frac{D_h}{Dt}\left(\frac{\zeta_g + f}{h}\right) = 0$$

Phase and Group Velocity (1D)

$$v_p = \frac{\nu}{k} \qquad v_g = \frac{\partial \nu}{\partial k}$$

Equations of Motion

$$\begin{aligned} \frac{d\rho}{dt} &= -\rho\nabla\cdot\mathbf{v} \\ \frac{du}{dt} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2\Omega v\sin\phi - 2\Omega w\cos\phi \\ \frac{dv}{dt} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2\Omega u\sin\phi \\ \frac{dw}{dt} &= -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + 2\Omega u\cos\phi \\ \frac{c_p}{T}\frac{dT}{dt} - \frac{R_d}{p}\frac{dp}{dt} &= \frac{J}{T} \\ p &= \rho R_d T \end{aligned}$$

Hydrostatic Equations of Motion

$$\begin{aligned} \frac{D\mathbf{v}_h}{Dt} + f\mathbf{k}\times\mathbf{v}_h = -\nabla_p\Phi \\ \frac{\partial\Phi}{\partial p} = -\frac{R_d T}{p} \\ \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)_p + \frac{\partial\omega}{\partial p} = 0 \\ \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} - S_p\omega = \frac{J}{c_p} \\ p = \rho R_d T \end{aligned}$$

Quasi-Geostrophic Equations

$$\begin{aligned} \frac{D_g\mathbf{v}_g}{Dt} = -f_0\mathbf{k}\times\mathbf{v}_a - \beta y\mathbf{k}\times\mathbf{v}_g \\ \mathbf{v}_g = \frac{1}{f_0}\mathbf{k}\times\nabla\Phi \qquad \left(\zeta_g = \frac{1}{f_0}\nabla^2\Phi\right) \\ \frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} + \frac{\partial\omega}{\partial p} = 0 \\ \left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\left(-\frac{\partial\Phi}{\partial p}\right) - \sigma\omega = \frac{\kappa J}{p} \end{aligned}$$

Shallow Water Equations

$$\begin{aligned} \frac{du}{dt} - fv + g\frac{\partial}{\partial x}(h + h_t) &= 0 \\ \frac{dv}{dt} + fu + g\frac{\partial}{\partial y}(h + h_t) &= 0 \\ \frac{dh}{dt} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) &= 0 \end{aligned}$$

Energy (per unit area)

Internal	$I = \int_0^\infty \rho c_v T dz$
Potential	$P = \int_0^\infty \rho g z dz$
Total potential	$TPE = I + P$
Kinetic	$K = \int_0^\infty \frac{\rho}{2}\lvert\mathbf{v}\rvert^2 dz$

11 Numerical Methods

Finite-Difference Formulas (Centered)

Difference	Formula	Error term
$f'(x_i)$	$\dfrac{f_{i+1} - f_{i-1}}{2h}$	$-\frac{1}{6}h^2 f^{(3)}$
	$\dfrac{-f_{i+2} + 8f_{i+1} - 8f_{i-1} + f_{i-2}}{12h}$	$\frac{1}{30}h^4 f^{(5)}$
	$\dfrac{f_{i+3} - 9f_{i+2} + 45f_{i+1} - 45f_{i-1} + 9f_{i-2} - f_{i-3}}{60h}$	$-\frac{1}{140}h^6 f^{(7)}$
$f''(x_i)$	$\dfrac{f_{i+1} - 2f_i + f_{i-1}}{h^2}$	$-\frac{1}{12}h^2 f^{(4)}$
	$\dfrac{-f_{i+2} + 16f_{i+1} - 30f_i + 16f_{i-1} - f_{i-2}}{12h^2}$	$\frac{1}{90}h^4 f^{(6)}$
	$\dfrac{2f_{i+3} - 27f_{i+2} + 270f_{i+1} - 490f_i + 270f_{i-1} - 27f_{i-2} + 2f_{i-3}}{180h^2}$	$-\frac{1}{560}h^6 y^{(8)}$
$f^{(3)}(x_i)$	$\dfrac{f_{i+2} - 2f_{i+1} + 2f_{i-1} - f_{i-2}}{2h^3}$	$-\frac{1}{4}h^2 f^{(5)}$
	$\dfrac{-f_{i+3} + 8f_{i+2} - 13f_{i+1} + 13f_{i-1} - 8f_{i-2} + f_{i-3}}{8h^3}$	$\frac{7}{120}h^4 f^{(7)}$
$f^{(4)}(x_i)$	$\dfrac{f_{i+2} - 4f_{i+1} + 6f_i - 4f_{i-1} + f_{i-2}}{h^4}$	$-\frac{1}{6}h^2 f^{(6)}$
	$\dfrac{-f_{i+3} + 12f_{i+2} - 39f_{i+1} + 56f_i - 39f_{i-1} + 12f_{i-2} - f_{i-3}}{6h^4}$	$\frac{7}{240}h^4 y^{(8)}$

Finite-Difference Formulas (Uncentered)

Difference	Formula	Error term	Formula	Error term
$f'(x_i)$	$\dfrac{f_{i+1} - f_i}{h}$	$-\frac{1}{2}hf^{(2)}$	$\dfrac{f_i - f_{i-1}}{h}$	$\frac{1}{2}hf^{(2)}$
	$\dfrac{-f_{i+2} + 4f_{i+1} - 3f_i}{2h}$	$\frac{1}{3}h^2 f^{(3)}$	$\dfrac{3f_i - 4f_{i-1} + f_{i-2}}{2h}$	$\frac{1}{3}h^2 f^{(3)}$
	$\dfrac{2f_{i+3} - 9f_{i+2} + 18f_{i+1} - 11f_i}{6h}$	$-\frac{1}{4}h^3 f^{(4)}$	$\dfrac{11f_i - 18f_{i-1} + 9f_{i-2} - 2f_{i-3}}{6h}$	$\frac{1}{4}h^3 f^{(4)}$
	$\dfrac{-f_{i+2} + 6f_{i+1} - 3f_i - 2f_{i-1}}{6h}$	$\frac{1}{12}h^3 f^{(4)}$	$\dfrac{2f_{i+1} + 3f_i - 6f_{i-1} + f_{i+2}}{6h}$	$-\frac{1}{12}h^3 f^{(4)}$
	$\dfrac{-3f_{i+4} + 16f_{i+3} - 36f_{i+2} + 48f_{i+1} - 25f_i}{12h}$	$\frac{1}{5}h^4 f^{(5)}$	$\dfrac{25f_i - 48f_{i-1} + 36f_{i-2} - 16f_{i-3} + 3f_{i-4}}{12h}$	$\frac{1}{5}h^4 f^{(5)}$
	$\dfrac{f_{i+3} - 6f_{i+2} + 18f_{i+1} - 10f_i - 3f_{i-1}}{12h}$	$-\frac{1}{20}h^4 f^{(5)}$	$\dfrac{3f_{i+1} + 10f_i - 18f_{i-1} + 6f_{i-2} - f_{i+3}}{12h}$	$-\frac{1}{20}h^4 f^{(5)}$
$f''(x_i)$	$\dfrac{f_{i+2} - 2f_{i+1} + f_i}{h^2}$	$-hf^{(3)}$	$\dfrac{f_i - 2f_{i-1} + f_{i-2}}{h^2}$	$hf^{(3)}$
	$\dfrac{-f_{i+3} + 4f_{i+2} - 5f_{i+1} + 2f_i}{h^2}$	$\frac{11}{12}h^2 f^{(4)}$	$\dfrac{2f_i - 5f_{i-1} + 4f_{i-2} - f_{i-3}}{h^2}$	$\frac{11}{12}h^2 f^{(4)}$
	$\dfrac{11f_{i+4} - 56f_{i+3} + 114f_{i+2} - 104f_{i+1} + 35f_i}{12h^2}$	$-\frac{5}{6}h^3 f^{(5)}$	$\dfrac{35f_i - 104f_{i-1} + 114f_{i-2} - 56f_{i-3} + 11f_{i-4}}{12h^2}$	$\frac{5}{6}h^3 f^{(5)}$
	$\dfrac{-f_{i+3} + 4f_{i+2} + 6f_{i+1} - 20f_i + 11f_{i-1}}{12h^2}$	$\frac{1}{12}h^3 f^{(5)}$	$\dfrac{11f_{i+1} - 20f_i + 6f_{i-1} + 4f_{i-2} - f_{i-3}}{12h^2}$	$-\frac{1}{12}h^3 f^{(5)}$
$f^{(3)}(x_i)$	$\dfrac{f_{i+3} - 3f_{i+2} + 3f_{i+1} - f_i}{h^3}$	$-\frac{3}{2}hf^{(4)}$	$\dfrac{f_i - 3f_{i-1} + 3f_{i-2} - f_{i-3}}{h^3}$	$\frac{3}{2}hf^{(4)}$
	$\dfrac{f_{i+2} - 3f_{i+1} + 3f_i - f_{i-1}}{h^3}$	$-\frac{1}{2}hf^{(4)}$	$\dfrac{f_{i+1} - 3f_i + 3f_{i-1} - f_{i-2}}{h^3}$	$\frac{1}{2}hf^{(4)}$
	$\dfrac{-3f_{i+4} + 14f_{i+3} - 24f_{i+2} + 18f_{i+1} - 5f_i}{2h^3}$	$\frac{7}{4}h^2 f^{(5)}$	$\dfrac{5f_i - 18f_{i-1} + 24f_{i-2} - 14f_{i-3} + 3f_{i-4}}{2h^3}$	$\frac{7}{4}h^2 f^{(5)}$
	$\dfrac{-f_{i+3} + 6f_{i+2} - 12f_{i+1} + 10f_i - 3f_{i-1}}{2h^3}$	$\frac{1}{4}h^2 f^{(5)}$	$\dfrac{3f_{i+1} - 10f_i + 12f_{i-1} - 6f_{i-2} + f_{i-3}}{2h^3}$	$\frac{1}{4}h^2 f^{(5)}$
$f^{(4)}(x_i)$	$\dfrac{f_{i+4} - 4f_{i+3} + 6f_{i+2} - 4f_{i+1} + f_i}{h^4}$	$-2hf^{(5)}$	$\dfrac{f_i - 4f_{i-1} + 6f_{i-2} - 4f_{i-3} + f_{i-4}}{h^4}$	$2hf^{(5)}$
	$\dfrac{f_{i+3} - 4f_{i+2} + 6f_{i+1} - 4f_i + f_{i-1}}{h^4}$	$-hf^{(5)}$	$\dfrac{f_{i+1} - 4f_i + 6f_{i-1} - 4f_{i-2} + f_{i-3}}{h^4}$	$hf^{(5)}$